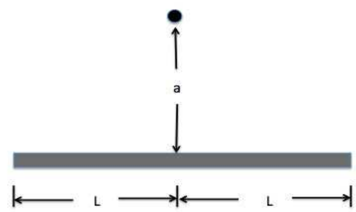
7. [9 points] A particle of mass m is positioned at a perpendicular distance a from the center of a rod of length $2L$ and constant mass density δ as shown below



The force of gravitational attraction F between the rod and the particle is given by

$$F = Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx.$$

a. [5 points] Does the force of gravitational attraction F approach infinity as the length of the rod goes to infinity? Justify your answer using the comparison test.

Solution:

$$F = Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx = 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx$$

$$\begin{aligned}
\lim_{L\to\infty} F &= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&\leq 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + \lim_{L\to\infty} 2Gm\delta a \int_{1}^{L} \frac{1}{(x^2)^{\frac{3}{2}}} dx \\
&= 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{x^3} dx.
\end{aligned}$$

The last integral converges since $\int_1^\infty \frac{1}{x^p}dx$ converges for $p = 3 > 1$.

b. [4 points] Consider the integral

$$I = \int_{1}^{\infty} \frac{1}{(a^2+x^2)^p} dx$$

i. Give a power function which, if integrated over $[1, \infty)$, will have the same convergence or divergence behavior as I.

Solution: $\frac{1}{(x^2)^p} = \frac{1}{x^{2p}}$

ii. For which values of p would you predict I is convergent? For which would I be divergent?

Solution: Convergent: Need $2p > 1$, hence $p > \frac{1}{2}$.

Divergent: Need $2p \leq 1$, hence $p \leq \frac{1}{2}$.