8. [12 points] Determine if the following integrals converge or diverge. Justify your answer. If you use the comparison test, be sure to show all your work.

a. [3 points] $\displaystyle\int_1^\infty \frac{1}{x+e^x}dx.$

Solution: Using the comparison

$$\frac{1}{x+e^x} \le \frac{1}{e^x},$$

we get convergence, since

$$\int_1^\infty \frac{1}{e^x}dx$$

converges.

b. [4 points] $\displaystyle\int_1^e \frac{1}{x(\ln x)^2}dx.$

Solution: This is improper because $\ln 1 = 0$, so there is an asymptote at $x = 1$. Here we use the substitution $u = \ln x$, so $du = \frac{1}{x}dx$, and we get

$$\int_1^e \frac{1}{x(\ln x)^2}dx = \int_0^1 \frac{1}{u^2}du.$$

The right hand side diverges by the p-test ($p = 2 > 1$).

c. [5 points] $\displaystyle\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx.$

Solution: Break this up into two integrals:

$$\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx = \int_{2\pi}^\infty \frac{x\cos^2 x}{x^3}dx + \int_{2\pi}^\infty \frac{1}{x^3}dx$$

The second integral converges by the p-test. For the first, we need to use another comparison:

$$\frac{x\cos^2 x}{x^3} \le \frac{1}{x^2}$$

so by comparison, the first integral also converges. The sum of two convergent improper integrals converges, so this integral converges.