9. [14 points] A machine produces copper wire, and occasionally there is a flaw at some point along the wire. The length x of wire produced between two consecutive flaws is a continuous variable with probability density function

$$f(x) = \begin{cases} c(1+x)^{-3} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work in order to receive full credit.

a. [5 points] Find the value of c.

Solution: Since $f(x)$ is a density function $\int_{-\infty}^{\infty} f(x)dx = 1$. Then

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} c(1+x)^{-3}dx = \lim_{b\to\infty} \int_0^b c(1+x)^{-3}dx$$
$$= \lim_{b\to\infty} \frac{-c}{2(1+x)^2}\Big|_0^b = \frac{c}{2} = 1$$

Hence $c = 2$.

b. [3 points] Find the cumulative distribution function $P(x)$ of the density function $f(x)$. Be sure to indicate the value of $P(x)$ for **all** values of x.

Solution:

$$P(x) = \int_{-\infty}^{x} f(t)dt = \int_0^x c(1+t)^{-3}dt = \frac{-c}{2(1+t)^2}\Big|_0^x = \frac{c}{2} - \frac{c}{2(1+x)^2} = 1 - \frac{1}{(1+x)^2}.$$

$$P(x) = \begin{cases} 1 - \frac{1}{(1+x)^2} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

c. [5 points] Find the mean length of wire between two consecutive flaws.

Solution:

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^\infty c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

$u = x \quad v' = (1+x)^{-3}$

$u' = 1 \quad v = \dfrac{-1}{2(1+x)^2}$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2}\Big|_0^b + \int_0^b \frac{1}{2(1+x)^2}dx$$

$$= c\lim_{b\to\infty} \frac{-x}{2(1+x)^2} - \frac{1}{2(1+x)}\Big|_0^b = \frac{c}{2} = 1.$$

or

$$\text{mean} = \int_{-\infty}^{\infty} xf(x)dx = \int_0^\infty c\frac{x}{(1+x)^3}dx = c\lim_{b\to\infty}\int_0^b \frac{x}{(1+x)^3}dx$$

$u = 1 + x$

$$= c\lim_{b\to\infty}\int_1^{b+1} \frac{u-1}{u^3}dx = c\lim_{b\to\infty}\int_1^{b+1} u^{-2} - u^{-3}dy$$

$$= c\lim_{b\to\infty} -u^{-1} + \frac{u^{-2}}{2}\Big|_1^{b+1} = \frac{c}{2} = 1.$$

d. [1 point] A second machine produces the same type of wire, but with a different probability density function (pdf). Which of the following graphs could be the graph of the pdf for the second machine? Circle all your answers.

Solution: The graph on the left upper corner can't be the density since$x$ is the distance between flaws, hence the probability density function can't be positive for $x < 0$.
The graph on the left lower corner can't be the density since the area under the curve for $x \geq 0$ is infinite (it has a positive horizontal asymptote).
The graph on the right upper corner can't be a density since it is negative on an interval.