6. [11 points]

a. [8 points] Use the **comparison method** to determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use.

i) $\displaystyle\int_1^\infty \frac{3+\sin(4x)}{\sqrt[3]{x}}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^{1/3}}$. Because $3+\sin(4x) \geq 2$, we know that

$$\frac{3+\sin(4x)}{x^{1/3}} \geq \frac{2}{x^{1/3}}.$$

By the p-test with $p = 1/3$, we know that $\int_1^\infty \frac{1}{x^{1/3}}dx$ diverges. Therefore, by the comparison method, we know that this integral diverges, too.

ii) $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}+x^2}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^2}$. Because $\sqrt{x} \geq 0$, we know that

$$\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{x^2}.$$

By the p-test with $p = 2$, we know that $\int_4^\infty \frac{1}{x^2}dx$ converges. Therefore, by the comparison test, $\int_4^\infty \frac{1}{\sqrt{x}+x^2}$ converges, too.

b. [3 points] For which values of p does the following integral converges?

$$\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$$

No justification is required.

Solution: If $p \leq 2$, the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{1}{4}$ for large values of x. Hence the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$ diverges.

If $p > 2$, then the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{x^2}{x^p} = \frac{1}{x^{p-2}}$ for large values of x. Then $\displaystyle\int_2^\infty \frac{1}{x^{p-2}}dx$ converges if $p > 3$ $(p-2 > 1)$. Therefore the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx$ converges for $p > 3$.