7. [15 points] A cone is filled with water up to a depth of $H_0$ m. At time $t = 0$, a valve at the bottom of the cone is opened. Water leaks out of the cone through the opened valve. Let $H(t)$ be the depth of the water (in m) in the cone at time $t$ (in hours). The function $H(t)$ satisfies the differential equation

$$\frac{dH}{dt} = \frac{k}{H^{3/2}}$$

a. [2 points]

What must be the sign and units of $k$?

Solution: The sign of $k$ is negative, because the water is dripping out. Because $dH/dt$ is in meters per hour and $H^{3/2}$ is in m$^{3/2}$, we know that $k$ must have units m$^{5/2}$ per hour.

b. [7 points] Find a formula for $H(t)$. Your formula should include $k$ and $H_0$

Solution: We use separation of variables.

$$\int H^{3/2}dH = \int kdt$$

$$\frac{H^{5/2}}{5/2} = kt + C_0$$

$$H^{5/2} = \frac{5}{2}kt + C_1$$

$$H = \left(\frac{5}{2}kt + C_1\right)^{2/5}$$

Since

$$H_0 = H(0) = C_1^{2/5}$$

we know that $C_1 = H_0^{5/2}$. Therefore,

$$H(t) = \left(\frac{5}{2}kt + H_0^{5/2}\right)^{2/5}$$

This problem continues on the next page.
Problem 7 continued

c. [4 points] If the cone is filled with water up to a depth of 4 m at \( t = 0 \). What should the value of \( k \) be in order for the cone to be empty after an hour? Show all your work.

Solution: We know \( H_0 = 4 \). Therefore, the equation is \((5/2kt + 32)^{2/5} = H(t)\). Setting this equal to zero, we have

\[
\left( \frac{5}{2} k + 32 \right)^{2/5} = 0
\]

So we want to solve

\[
\frac{5}{2} k + 32 = 0 \Rightarrow k = -\frac{64}{5}.
\]

d. [2 points] Does the differential equation satisfied by \( H \) have equilibrium solutions? If it does, find them.

Solution: No