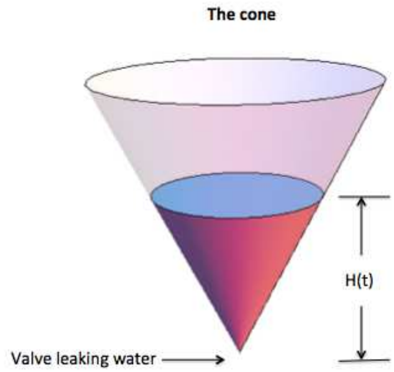
7. [15 points] A cone is filled with water up to a depth of H_0 m. At time $t = 0$, a valve at the bottom of the cone is opened. Water leaks out of the cone through the opened valve. Let $H(t)$ be the depth of the water (in m) in the cone at time t (in hours).

The function $H(t)$ satisfies the differential equation

$$\frac{dH}{dt} = \frac{k}{H^{\frac{3}{2}}}$$



a. [2 points]
What must be the sign and units of k?

Solution: The sign of k is negative, because the water is dripping out. Because dH/dt is in meters per hour and $h^{3/2}$ is in $\mathrm{m}^{3/2}$, we know that k must have units $\mathrm{m}^{5/2}$ per hour.

b. [7 points] Find a formula for $H(t)$. Your formula should include k and H_0

Solution: We use separation of variables.

$$\int H^{3/2} dH = \int k dt$$

$$\frac{H^{5/2}}{5/2} = kt + C_0$$

$$H^{5/2} = \frac{5}{2}kt + C_1$$

$$H = \left(\frac{5}{2}kt + C_1\right)^{2/5}$$

Since

$$H_0 = H(0) = C_1^{2/5}$$

we know that $C_1 = H_0^{5/2}$. Therefore,

$$H(t) = \left(\frac{5}{2}kt + H_0^{5/2}\right)^{2/5}.$$

This problem continues on the next page.

Problem 7 continued

c. [4 points] If the cone is filled with water up to a depth of 4 m at $t = 0$. What should the value of k be in order for the cone to be empty after an hour? Show all your work.

Solution: We know $H_0 = 4$. Therefore, the equation is $(5/2kt + 32)^{2/5} = H(t)$. Setting this equal to zero, we have

$$\left(\frac{5}{2}k + 32\right)^{2/5} = 0$$

So we want to solve

$$\frac{5}{2}k + 32 = 0 \Rightarrow k = -\frac{64}{5}.$$

d. [2 points] Does the differential equation satisfied by H have equilibrium solutions? If it does, find them.

Solution: No