6. [9 points] An extremely sleepy graduate student is grading Math 116 exams. She has been drinking coffee all day, but it just is not enough. She hooks up a caffeine drip that delivers caffeine into her body at a constant rate of 170 mg/hr. The amount of caffeine in her body decays at a rate proportional to the current amount of caffeine in her body. The half-life of caffeine in her body is 6 hours.

a. [4 points] Using the blank provided, write a differential equation which models the scenario described above. Use $Q(t)$ for the amount of caffeine in the graduate student's body, measured in mg, t for hours after she hooked up the caffeine drip, and $k > 0$ for the constant of proportionality.

Solution: The rate that the amount of caffeine in the graduate student's body is changing over time should be the rate that caffeine is entering their body minus the rate that caffeine is leaving their body. The rate that caffeine is entering the graduate students body is a constant 170 mg/hr. The rate that caffeine is leaving the graders body is proportional to the current amount, so it is kQ mg/hr. Putting all this together gives us the equation written below.

$$\frac{dQ}{dt} = \underline{\qquad 170 - kQ \qquad}$$

b. [5 points] Use the half-life of caffeine to determine the constant of proportionality.

Solution: We know that the amount of caffeine in the graduate student's body decays exponentially with decay rate k. If C_0 is the initial amount of caffeine, then a half-life of 6 hours means that

$$\frac{1}{2}C_0 = C_0 e^{-k6}.$$

Solving for k gives us that $k = -\frac{1}{6}\ln\left(\frac{1}{2}\right)$.