9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

Solution: $t \geq 1 \Rightarrow \frac{1}{t} \leq 1 \Rightarrow \cos(\frac{1}{t}) \geq \cos(1)$ because the function $F(x) = \cos x$ is decreasing in the interval $[0, 1]$. Therefore,

$$\frac{\cos(\frac{1}{t})}{\sqrt{t}} \geq \frac{\cos(1)}{\sqrt{t}}$$

The improper integral

$$\int_1^\infty \frac{\cos(1)}{\sqrt{t}}\, dt = \cos(1)\int_1^\infty \frac{1}{\sqrt{t}}\, dt$$

diverges by the p-test since $p = \frac{1}{2} \leq 1$. So the integral

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt$$

diverges by the comparison test (notice that $\cos(1) > 0$).

b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}}$$

Solution: Notice that by (a), this is $\frac{\infty}{\infty}$. We use L'Hopital's rule along with the 2nd Fundamental Theorem in the numerator:

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\, dt}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$$