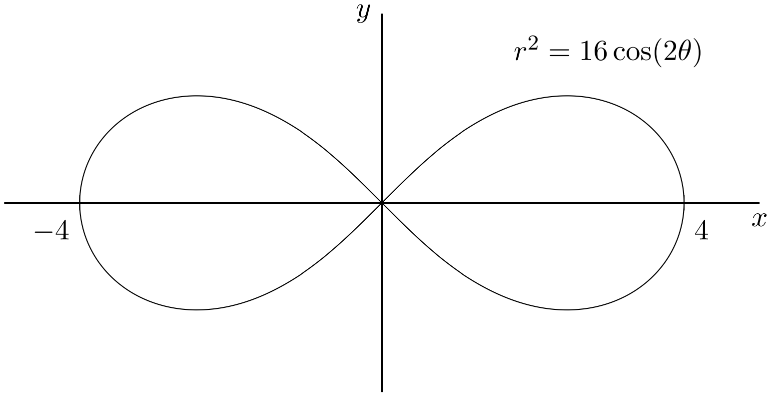
2. [12 points] Chancelor was doodling in his coloring book one Sunday afternoon when he drew an infinity symbol, or lemniscate. The picture he drew is the polar curve $r^2 = 16\cos(2\theta)$, which is shown on the axes below. (The axes are measured in inches.)



a. [4 points] Chancelor decides to color the inside of the lemniscate red. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he has to fill in with red.

Solution: Notice that we are given a formula for r^2 instead of r. Using symmetry, we will calculate the area in one quarter of the lemniscate and multiply by 4. To do this we will integrate from $\theta = 0$ to $\theta = \alpha$ where α is the smallest positive number for which $16\cos(2\alpha) = 0$. This gives $\alpha = \pi/4$. Using the formula for area inside a polar curve we see that the area is equal to $4 \cdot \frac{1}{2}\int_0^{\pi/4} 16\cos(2\theta)\, d\theta$ square inches.

b. [4 points] He decides he wants to outline the right half (the portion to the right of the y-axis) of the lemniscate in blue. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total length, in inches, of the outline he must draw in blue.

Solution: The portion of the lemniscate on the right of the y-axis corresponds to $-\pi/4 < \theta < \pi/4$. Notice that $\cos(2\theta) > 0$ for these angles.

Implicitly differentiating $r^2 = 16\cos(2\theta)$ (or directly differentiating $r = 4(\cos(2\theta))^{1/2}$), we find that $\frac{dr}{d\theta} = \frac{-4\sin(2\theta)}{\sqrt{\cos(2\theta)}}$ so $\left(\frac{dr}{d\theta}\right)^2 = \frac{16\sin^2(2\theta)}{\cos(2\theta)}$.
Then using the arc length formula for polar coordinates we see that the length of the blue outline will be $\int_{-\pi/4}^{\pi/4} \sqrt{16\cos(2\theta) + \frac{16\sin^2(2\theta)}{\cos(2\theta)}}\, d\theta$ inches.

c. [4 points] Chancelor draws another picture of the same lemniscate, but this time also draws a picture of the circle $r = 2\sqrt{2}$. He would like to color the area that is inside the lemniscate but outside the circle purple. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he must fill in with purple.

Solution: The circle and lemniscate intersect on the right side of the y-axis when $16\cos(2\theta) = 8$ or $\cos(2\theta) = \frac{1}{2}$. This gives angles $\theta_1 = -\pi/6$ and $\theta_2 = \pi/6$.

We first find the area between the curves on the right side using the formula for the area between polar curves and then multiply the by 2.
The resulting total area is $2 \cdot \frac{1}{2}\int_{-\pi/6}^{\pi/6} (16\cos(2\theta) - 8)\, d\theta$ square inches.