7. [10 points] Fatimah begins to make herself a cup of tea to soothe her sore throat. As soon as the water boils, her telephone rings and she answers it. While she is on the phone, the hot water begins to cool. The temperature of the hot water (in °C) is given by a function $H = H(t)$, where t is measured in minutes after the water boils. Suppose $H(t)$ satisfies the differential equation

$$\frac{dH}{dt} = k(H-20)t^2$$

for some constant k.
(Note: This is **not** Newton's Law of Cooling. Her teapot has unusual thermal properties.)

Assume that the water boils at 100°C.

a. [1 point] What are the units of the constant k?

Solution:

$$1/\mathrm{min}^3$$

b. [6 points] Use separation of variables to find a formula for $H(t)$ by hand. Your formula may involve k, but should not involve any other unknown constants. Be sure to show your work.

Solution: Separating variables gives

$$\frac{dH}{H-20} = kt^2\,dt.$$

Integrating we find,

$$\int \frac{dH}{H-20} = \int kt^2\,dt$$
$$\ln|H-20| = \frac{kt^3}{3} + C,$$

for some constant C.
Exponentiation of both sides of this equation gives

$$|H-20| = e^{kt^3/3+C} = Ae^{kt^3/3},$$

where $A = e^C$ is a positive real number. Then we have

$$H - 20 = Be^{kt^3/3}$$
$$H = 20 + Be^{kt^3/3}$$

where $B = \pm A$ is a nonzero real number. Note that $H = 20$ (corresponding to $B = 0$) is an equilibrium solution, so the general solution is $H = 20 + Be^{kt^3/3}$ where B is any real number.
Since the initial condition is $H(0) = 100$, we find that $100 = 20 + B$, so $B = 80$, and the particular solution is $H = 20 + 80e^{kt^3/3}$.

c. [3 points] Suppose that 3 minutes after boiling, the temperature of the water is 60°C. What is the value of k?

Solution: This new information tells us that $H(3) = 60$. Plugging this in to our solution above, we find that $60 = 20 + 80e^{k3^3/3} = 20 + 80e^{9k}$.
So,

$$80e^{9k} = 40$$
$$e^{9k} = 0.5$$
$$k = \ln(0.5)/9.$$