9. [12 points] Determine whether the following integrals converge or diverge.
If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided.
If the integral diverges, circle "diverges" and justify your answer.
In either case, **you must show all your work and indicate any theorems you used to conclude convergence or divergence of the integrals.** Any direct evaluation of integrals must be done **without using a calculator.**

a. [6 points] $\displaystyle\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$

[✓ Diverges] **Converges to** ______________

Solution: If the improper integral converges, we can rewrite it as a sum

$$\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx = \int_{-\infty}^{1} \frac{2}{(1+x^4)^{1/4}}\,dx + \int_{1}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx.$$

To show that the original improper integral diverges, is enough to show that the last improper integral above diverges.
We will use direct comparison. First, we note that for $x \geq 1$ $\dfrac{2}{(1+x^4)^{1/4}} \geq \dfrac{1}{x}$.
Additionally $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges as it is an integral of the form $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ with $p = 1 \leq 1$.
So by direct comparison, we see that the improper integral
$\displaystyle\int_1^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$
also diverges. Thus the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$ diverges.

b. [6 points] $\displaystyle\int_1^{e} \frac{x^3-3x^3\ln(x)-1}{x(x^3-1)^2}\,dx$ *Hint:* $\displaystyle\frac{d}{dx}\left(\frac{\ln(x)}{x^3-1}\right) = \frac{x^3-3x^3\ln(x)-1}{x(x^3-1)^2}$

Diverges **[✓ Converges]** **to** $\displaystyle\frac{1}{e^3-1} - \frac{1}{3}$

Solution:

$$\int_1^{e} \frac{x^3-3x^3\ln(x)-1}{x(x^3-1)^2}\,dx = \lim_{b\to 1^+} \int_b^{e} \frac{x^3-3x^3\ln(x)-1}{x(x^3-1)^2} = \lim_{b\to 1^+} \frac{\ln(x)}{x^3-1}\Bigg|_b^e$$

$$= \lim_{b\to 1^+}\left(\frac{\ln(e)}{e^3-1} - \frac{\ln(b)}{b^3-1}\right) = \frac{1}{e^3-1} - \lim_{b\to 1^+}\frac{\ln(b)}{b^3-1}.$$

As this last limit has the indeterminate form $0/0$ we can apply L'Hopital's rule.

$$\lim_{b\to 1^+}\frac{\ln(b)}{b^3-1} = \lim_{b\to 1^+}\frac{1/b}{3b^2} \text{ (by L'Hopital's rule)}$$
$$= \lim_{b\to 1^+}\frac{1}{3b^3} = \frac{1}{3}.$$

So we find that the improper integral $\displaystyle\int_1^{e} \frac{x^3-3x^3\ln(x)-1}{x(x^3-1)^2}\,dx$ converges to $\dfrac{1}{e^3-1} - \dfrac{1}{3}$.