10. [10 points] Suppose $Q(x)$ is the cumulative distribution function (cdf) for a variable $x$, such that

$$Q(x) = \begin{cases} a & \text{for } x \leq 0 \\ b - e^{-cx} & \text{for } x > 0, \end{cases}$$

and the median value of $x$ is 2.

a. [2 points] Let $q(x)$ be the probability density function for $x$. Write a formula for $q(x)$, assuming $q(0) = 0$.

Answer: $q(x)$ is an antiderivative of $q(x)$, so we can differentiate $Q(x)$ to get

$$q(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ ce^{-cx} & \text{for } x > 0 \end{cases}$$

b. [4 points] Set up, but do not evaluate, an expression involving one or more integrals that represents the mean value of $x$. Your answer may contain $a$, $b$, or $c$, but should not contain any function names (such as $Q$ or $q$).

Answer: $\int_{0}^{\infty} cx e^{-cx} \, dx$

c. [4 points] Find the values of $a$, $b$, and $c$. Justify your answers, and write them in exact form. Remember that the median value of $x$ is 2.

Solution: We know $\lim_{x \to -\infty} Q(x) = a = 0$ and $\lim_{x \to \infty} Q(x) = b = 1$. Finally, we have

$$Q(2) = 1/2$$
$$1 - e^{2c} = 1/2$$
$$e^{-2c} = 1/2$$
$$-2c = \ln(1/2)$$
$$c = -\ln(1/2)/2 = \ln(2)/2$$

Answer: $a = 0$, $b = 1$, $c = \ln(2)/2$