4. [8 points] Consider the power series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{3^n n\sqrt{\ln(n)}}.$$

The radius of convergence of this power series is $R = 3$. Determine the interval of convergence for this power series and fully justify the convergence or divergence at the endpoints. **You may assume $R = 3$ without justification.**

Solution: Since the radius is 3 and the center is 1, we know that the series must converge for all values of x between $1 - 3 = -2$ and $1 + 3 = 4$ and diverge for $x < -2$ and $x > 4$. Now we check convergence at each of the endpoints.

When $x = -2$ we get the series

$$\sum_{n=2}^{\infty} \frac{(-3)^n}{3^n n\sqrt{\ln(n)}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}.$$

This is an alternating series, and since $\dfrac{1}{n\sqrt{\ln(n)}}$ is decreasing and has limit 0, the series converges by the alternating series test.

When $x = 4$ we get the series

$$\sum_{n=2}^{\infty} \frac{3^n}{3^n n\sqrt{\ln(n)}} = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}.$$

Using the integral test, we consider the improper integral

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \int_{\ln(2)}^{\infty} \frac{1}{\sqrt{u}}\,du.$$

This integral diverges by the p-test ($p = 1/2$). Therefore the interval of convergence is $[-2, 4)$.

Answer: Interval of convergence = $[-2, 4)$