6. [12 points] Determine whether the following series converge or diverge.
Fully justify your answer. Show all work and indicate any convergence tests used.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: Since

$$\frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}} \approx \frac{1}{n^2},$$

we use the limit comparison test comparing with $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+n\cos(n)}{\sqrt{n^8-n+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4 + n^3\cos(n)}{\sqrt{n^8 - n + 1}} = 1.$$

We can see this is true by using domination arguments: the numerator is dominated by n^4, while $\sqrt{n^8 - n + 1}$ is dominated by $\sqrt{n^8}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test ($p = 2$), our original series converges by the LCT.

b. [6 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: The terms in this series are not positive, but it is also not an alternating series. We will consider the series

$$\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right| = \sum_{n=0}^{\infty} \frac{|\sin(n)|}{e^n}. \tag{1}$$

Since $|\sin(n)| \le 1$ we have

$$\frac{|\sin(n)|}{e^n} \le \frac{1}{e^n}$$

for all $n \ge 0$. The larger series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by the geometric series test since the common ratio $1/e$ is less than 1. (Note that there are many other ways to show that this series converges.) Therefore $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges by comparison. Since $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges, our original series $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ converges absolutely, and, specifically, must itself converge (this is sometimes called the absolute convergence test).