8. [7 points] Consider the integral

$$\int_1^\infty \frac{e^{rx}}{x}\,dx,$$

where r is a constant.

a. [3 points] Show that this integral converges for $r < 0$. **Show all work and indicate any convergence tests used.**

Solution: We know that $\frac{e^{rx}}{x} \le e^{rx}$ for all $x \ge 1$.

Further, when $r < 0$, we know that $\int_1^\infty e^{rx}\,dx$ converges by exponential decay test.

Therefore, by (direct) comparison test, $\int_1^\infty \frac{e^{rx}}{x}\,dx$ converges.

b. [4 points] Show that the integral diverges for $r \ge 0$. **Show all work and indicate any convergence tests used.**

Solution: Now, when $r \ge 1$, we know that $e^{rx} \ge 1$ for all $x \ge 1$, so $\frac{e^{rx}}{x} \ge \frac{1}{x}$.

$\int_1^\infty \frac{dx}{x}$ diverges by p-test with $p = 1$.

Therefore, by comparison, $\int_1^\infty \frac{e^{rx}}{x}\,dx$ diverges.

Alternative solution:

For $r > 0$, $\lim_{x\to\infty} \frac{e^{rx}}{x} = \infty$. Since the integrand approaches infinity, the integral diverges.

This still leaves the $r = 0$ case. In this case, $\frac{e^{rx}}{x} = \frac{1}{x}$, so the integral diverges by p-test with $p = 1$.