3. [10 points] For the following parts, show all of your work and indicate any theorems you used to conclude convergence or divergence of the integrals. Any direct evaluation of integrals must be done without using a calculator.

a. [5 points] Determine whether the improper integral $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx$ converges or diverges.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: Write

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \int_0^1 \frac{1}{x^3+1}\,dx + \int_1^\infty \frac{1}{x^3+1}\,dx,$$

and observe that the first integral on the right-hand side is proper. For the second integral, note that

$$\frac{1}{x^3+1} \le \frac{1}{x^3}$$

for $1 \le x < \infty$, and the integral $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges by p-test with $p = 3 > 1$. We conclude that $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ also converges by comparison test, therefore the original integral $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ also converges.

b. [5 points] Give full justification of the following equation. Write out all steps by hand, including any integration techniques used.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = 1$$

Solution:

$$\begin{aligned}
\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx \\
&= \lim_{b\to\infty} \left(-\frac{\ln(x)}{x}\Bigg|_1^b + \int_1^b \frac{1}{x^2}\,dx \right), \qquad \text{by IBP with } u = \ln(x), dv = \frac{1}{x^2}dx, \\
&= \lim_{b\to\infty} \left(-\frac{\ln(b)}{b} - \frac{1}{b} + 1 \right) \\
&= -\lim_{b\to\infty} \left(\frac{\ln(b)}{b} \right) + 1 \\
&\overset{H}{=} -\lim_{b\to\infty} \left(\frac{1/b}{1} \right) + 1 \\
&= 1.
\end{aligned}$$