5. [12 points] Determine whether each of the following series converges, conditionally converges, or diverges. Fully justify your answer. Include any convergence tests used.

a. [6 points] \[ \sum_{n=1}^{\infty} ne^{-2n} \]

Circle one: Absolutely convergent Conditionally convergent Divergent

Justification:

Solution: Since \( ne^{-n} < 1 \) for all \( n \geq 1 \),

\[ ne^{-2n} = (ne^{-n})e^{-n} \leq e^{-2n} \]

and the series \( \sum_{n=1}^{\infty} e^{-n} \) converges since it is a geometric series with common ratio \( e^{-1} \)

whose magnitude is strictly less than 1, so \( \sum_{n=1}^{\infty} ne^{-2n} \) converges by comparison test.

Since \( |ne^{-2n}| = ne^{-2n} \), the series \( \sum_{n=1}^{\infty} ne^{-2n} \) is absolutely convergent.

b. [6 points] \[ \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n} + 2}{\sqrt{n^3 + 1}} \]

Circle one: Absolutely convergent Conditionally convergent Divergent

Justification:

Solution: Since \( \frac{\sqrt{n}+2}{\sqrt{n^3+1}} \) is decreasing and has limit zero, \( \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n} + 2}{\sqrt{n^3 + 1}} \) converges by the alternating series test.

On the other hand, since

\[ \frac{\sqrt{n} + 2}{\sqrt{n^3 + 1}} \geq \frac{\sqrt{n}}{\sqrt{n^3 + n^3}} = \frac{1}{2n}, \]

and the series \( \sum_{n=1}^{\infty} \frac{1}{2n} \) diverges by \( p \)-test with \( p = 1 \), we conclude by comparison test that \( \sum_{n=1}^{\infty} \frac{\sqrt{n} + 2}{\sqrt{n^3 + 1}} \) diverges. Therefore, the original series is only conditionally convergent.