5. [12 points] Determine whether each of the following series converges, conditionally converges, or diverges. Fully justify your answer. Include any convergence tests used.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$

Circle one: $\boxed{\textbf{Absolutely convergent}}$ **Conditionally convergent** **Divergent**

Justification:

Solution: Since $ne^{-n} < 1$ for all $n \geq 1$,

$$ne^{-2n} = (ne^{-n})e^{-n} \leq e^{-n}$$

and the series $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ converges since it is a geometric series with common ratio e^{-1} whose magnitude is strictly less than 1, so $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ converges by comparison test.

Since $|ne^{-2n}| = ne^{-2n}$, the series $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ is absolutely convergent.

b. [6 points] $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$

Circle one: **Absolutely convergent** $\boxed{\textbf{Conditionally convergent}}$ **Divergent**

Justification:

Solution: Since $\frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ is decreasing and has limit zero, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ converges by the alternating series test.

On the other hand, since

$$\frac{\sqrt{n}+2}{\sqrt{n^3+1}} \geq \frac{\sqrt{n}}{\sqrt{n^3+n^3}} = \frac{1}{2n},$$

and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by p-test with $p = 1$, we conclude by comparison test that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ diverges. Therefore, the original series is only conditionally convergent.