7. [10 points]

a. [5 points] Determine the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2(2n)!}{2^n(n!)^2}x^{2n}$.

Solution: Compute

$$\lim_{n\to\infty} \frac{\dfrac{(n+1)^2(2(n+1))!}{2^{n+1}((n+1)!)^2}|x|^{2(n+1)}}{\dfrac{n^2(2n)!}{2^n(n!)^2}|x|^{2n}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}\frac{(2n+2)!}{(2n)!}\frac{2^n}{2^{n+1}}\frac{(n!)^2}{((n+1)!)^2}\frac{|x|^{2n+2}}{|x|^{2n}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2n^2}|x|^2$$

$$= 2|x|^2.$$

By ratio test, the power series converges for

$$2|x|^2 < 1 \iff |x|^2 < \frac{1}{2} \iff |x| < \frac{1}{\sqrt{2}}.$$

Answer: $\frac{1}{\sqrt{2}}$

b. [5 points] You do not need to justify your answers below.
Suppose C_n is a sequence such that the following are true:

- C_n is a monotone decreasing sequence
- C_n converges to 0
- $\displaystyle\sum_{n=0}^{\infty} C_n$ diverges
- $\displaystyle\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$ has radius of convergence 4.

What is the center of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

Answer: -3

What are the endpoints of the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

Answer: Left endpoint at $a =$ -7

Right endpoint at $b =$ 1

Let a and b be the left and right endpoints of the interval of convergence you found above.

Which of the following could be the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

$[a, b]$ $(a, b]$ (a, b) [✓ $[a, b)$]