8. [12 points] Let $p(x)$ be the probability density function given by

$$p(x) = \begin{cases} \dfrac{c}{x^2} & \text{for } x \leq -1 \\ c & \text{for } -1 < x < 1 \\ \dfrac{c}{x^2} & \text{for } x \geq 1. \end{cases}$$

for some value c.

a. [5 points] Find the value of c. Justify your answer. Any integrals must be computed by hand.

Solution: We know that for any pdf $p(x)$, we must have $\int_{-\infty}^{\infty} p(x)\,dx = 1$. In this case, that means

$$\int_{-\infty}^{-1} c/x^2\,dx + \int_{-1}^{1} c\,dx + \int_{1}^{\infty} c/x^2\,dx = 1.$$

Changing to limits gives

$$\lim_{a\to-\infty} \int_{a}^{-1} c/x^2\,dx + \int_{-1}^{1} c\,dx + \lim_{b\to\infty} \int_{1}^{b} c/x^2\,dx = 1.$$

An antiderivative of c/x^2 is $-c/x$, so the line above can be written as

$$\lim_{a\to-\infty} (-c/(-1) - (-c/a)) + 2c + \lim_{b\to\infty} ((-c/b) - (-c/1)) = 1$$

Since $\lim_{a\to-\infty} -c/a = \lim_{b\to\infty} -c/b = 0$, we have $4c = 1$ or $c = 1/4$.

Answer: $c =$ **1/4**

b. [4 points] Find a piecewise-defined formula for $P(x)$, the cumulative density function for x.

Solution: The CDF is an antiderivative of the PDF. We need to choose correct constants to add on each piece so that $P(x)$ is increasing, $\lim_{x\to-\infty} P(x) = 0$, and $\lim_{x\to\infty} P(x) = 1$. This happens when

$$P(x) = \begin{cases} -\dfrac{c}{x} & \text{for } x \leq -1 \\ cx + 2c & \text{for } -1 < x < 1 \\ 1 - \dfrac{c}{x} & \text{for } x \geq 1. \end{cases}$$

c. [3 points] Show that there is no mean value of x.

Solution: We find the mean by finding $\int_{-\infty}^{\infty} xp(x)\,dx$. In this case, that means

$$\int_{-\infty}^{-1} c/x\,dx + \int_{-1}^{1} cx\,dx + \int_{1}^{\infty} c/x\,dx.$$

If any of these diverge, then the integral from $-\infty$ to ∞ also diverges, and $\int_{1}^{\infty} c/x\,dx$ diverges by p-test, $p = 1$.