2. [12 points] In order to build a settlement on the island, intruders start cutting down trees at the forest, cutting the trees into logs, and putting the logs in a pile. Let \( A_n \) be the number of logs they have in the pile at noon on the \( n \)-th day. The intruders have 100 logs in the pile at noon on the first day (so \( A_1 = 100 \)). Every day (between noon on one day and noon on the next day), the building team uses 10% of the logs in the pile, while the log-cutting team adds 20 logs to the pile immediately before noon.

a. [4 points] Find \( A_2 \) and \( A_3 \). You do not need to simplify your answers.

Solution:

\[
A_2 = 100 \cdot 0.9 + 20 \\
A_3 = (100 \cdot 0.9 + 20) \cdot 0.9 + 20 = 100 \cdot 0.9^2 + 20 + 20 \cdot 0.9
\]

b. [5 points] Find a closed form expression for \( A_n \). Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your closed form answer.

Solution: From observing the pattern from \( A_1, A_2 \) and \( A_3 \), we have

\[
A_n = 100 \cdot 0.9^{n-1} + (20 + 20 \cdot 0.9 + 20 \cdot 0.9^2 + \cdots + 20 \cdot 0.9^{n-2}) \\
= 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9}.
\]

Note that the term 100 \( \cdot 0.9^{n-1} \) is not part of the geometric series. There are \( n - 1 \) terms in the geometric series, so the exponent in the closed form is \( n - 1 \).

c. [3 points] How many logs will the intruders have in the pile in the long run?

Solution:

\[
\lim_{n \to \infty} 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9} = 0 + 20 \cdot \frac{1}{1 - 0.9} = 20 \cdot 10 = 200.
\]