5. [12 points] Another function $f(t)$ given by

$$f(t) = \begin{cases} \frac{t}{6} & \text{if } 0 < t \leq 2, \\ \frac{1}{3} & \text{if } 2 < t \leq 4, \\ 0 & \text{else.} \end{cases}$$

is the probability density function for the number of months $t$ that it will take the intruders to build the settlement.

a. [3 points] Find the probability that it will take the intruders between 1 and 2 months to build the settlement.

**Solution:** Probability that it will take the intruders between 1 and 2 months to build the settlement

$$= \int_{1}^{2} f(t) \, dt = \int_{1}^{2} \frac{t}{6} \, dt = \frac{2^2}{12} - \frac{1^2}{12} = \frac{3}{12} = \frac{1}{4}.$$

b. [5 points] Find the mean number of months it will take the intruders to build the settlement.

**Solution:**

$$\text{mean} = \int_{0}^{4} t f(t) \, dt$$

$$= \int_{0}^{2} t f(t) \, dt + \int_{2}^{4} t f(t) \, dt$$

$$= \int_{0}^{2} \frac{t^2}{6} \, dt + \int_{2}^{4} \frac{t}{3} \, dt$$

$$= \left( \frac{2^3}{18} - \frac{0^3}{18} \right) + \left( \frac{4^2}{6} - \frac{2^2}{6} \right)$$

$$= \frac{4}{9} + 2$$


c. [4 points] Find the median number of months it will take the intruders to build the settlement.
Solution: The area for the portion $2 < t \leq 4$ is $2/3$, which is more than $1/2$. Thus the
median is between 2 and 4.
Let $T$ be the median. Then

$$\int_T^4 f(t) \, dt = \frac{1}{2},$$

$$(4 - T)\frac{1}{3} = \frac{1}{2},$$

$$T = 4 - \frac{3}{2} = \frac{5}{2}.$$

Alternatively: We can set up with “the left half of the area is $1/2$”. Let $T$ be the
median.

$$\int_0^T f(t) \, dt = \frac{1}{2}.$$  

With the same analysis as above, we know that the median lies between 2 and 4. Thus
we split the integral.

$$\int_0^2 f(t) \, dt + \int_2^T f(t) \, dt = \frac{1}{2},$$

$$\int_0^2 \frac{t}{6} \, dt + \int_2^T \frac{1}{3} \, dt = \frac{1}{2},$$

$$\left(\frac{2^2}{12} - \frac{0^2}{12}\right) + \left(\frac{T}{3} - \frac{2}{3}\right) = \frac{1}{2},$$

$$T = \frac{5}{2}.$$