5. [12 points] Another function $f(t)$ given by

$$f(t) = \begin{cases} \frac{t}{6} & \text{if } 0 < t \leq 2, \\ \frac{1}{3} & \text{if } 2 < t \leq 4, \\ 0 & \text{else.} \end{cases}$$

is the probability density function for the number of months t that it will take the intruders to build the settlement.

a. [3 points] Find the probability that it will take the intruders between 1 and 2 months to build the settlement.

Solution: Probability that it will take the intruders between 1 and 2 months to build the settlement

$$= \int_1^2 f(t)\, dt = \int_1^2 \frac{t}{6}\, dt = \frac{2^2}{12} - \frac{1^2}{12} = \frac{3}{12} = \frac{1}{4}.$$

b. [5 points] Find the mean number of months it will take the intruders to build the settlement.

Solution:

$$\begin{aligned} \text{mean} &= \int_0^4 tf(t)\, dt \\ &= \int_0^2 tf(t)\, dt + \int_2^4 tf(t)\, dt \\ &= \int_0^2 \frac{t^2}{6}\, dt + \int_2^4 \frac{t}{3}\, dt \\ &= (\frac{2^3}{18} - \frac{0^3}{18}) + (\frac{4^2}{6} - \frac{2^2}{6}) \\ &= \frac{4}{9} + 2 \end{aligned}$$

c. [4 points] Find the median number of months it will take the intruders to build the settlement.

Solution: The area for the portion $2 < t \leq 4$ is $2/3$, which is more than $1/2$. Thus the median is between 2 and 4.
Let T be the median. Then

$$\int_T^4 f(t)\ dt = \frac{1}{2},$$
$$(4-T)\frac{1}{3} = \frac{1}{2},$$
$$T = 4 - \frac{3}{2} = \frac{5}{2}.$$

Alternatively: We can set up with "the left half of the area is $1/2$". Let T be the median.

$$\int_0^T f(t)\ dt = \frac{1}{2}.$$

With the same analysis as above, we know that the median lies between 2 and 4. Thus we split the integral.

$$\int_0^2 f(t)\ dt + \int_2^T f(t)\ dt = \frac{1}{2},$$
$$\int_0^2 \frac{t}{6}\ dt + \int_2^T \frac{1}{3}\ dt = \frac{1}{2},$$
$$(\frac{2^2}{12} - \frac{0^2}{12}) + (\frac{T}{3} - \frac{2}{3}) = \frac{1}{2},$$
$$T = \frac{5}{2}.$$