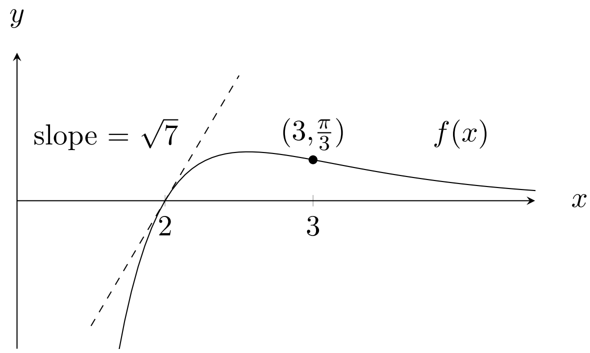
6. [12 points] Below is the graph of a function $f(x)$. The function $f(x)$ is positive for $x > 2$. The x-axis is a horizontal asymptote of $f(x)$ as $x \to \infty$. The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is $\sqrt{7}$.



a. [2 points] Compute $\displaystyle\lim_{x\to 2^+} \tan^{-1} \frac{1}{f(x)}$.

(*Hint:* Recall that $\displaystyle\lim_{z\to\infty} \tan^{-1} z = \frac{\pi}{2}$ and $\displaystyle\lim_{z\to-\infty} \tan^{-1} z = -\frac{\pi}{2}$.)

Solution: As $x \to 2^+$, $f(x)$ is approaching 0^+ from the graph. Hence $\dfrac{1}{f(x)}$ is going to $+\infty$, so

$$\lim_{x\to 2^+} \tan^{-1} \frac{1}{f(x)} = \frac{\pi}{2}.$$

b. [3 points] Compute $\displaystyle\lim_{x\to 2^-} \frac{f(x)}{e^x - e^2}$.

Solution: As $x \to 2^-$, both the numerator and the denominator is approaching 0. Hence we can apply L'Hopital's Rule.

$$\lim_{x\to 2^-} \frac{f(x)}{e^x - e^2} = \lim_{x\to 2^-} \frac{f'(x)}{e^x} = \frac{\sqrt{7}}{e^2}.$$

c. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation.**

$$\int_3^\infty \frac{f'(x)}{(f(x))^{2/3}}\, dx$$

Solution:

$$\int_3^{\infty} \frac{f'(x)}{(f(x))^{2/3}}\, dx = \lim_{b\to\infty} \int_3^{b} \frac{f'(x)}{(f(x))^{2/3}}\, dx$$

Apply substitution, with $w = f(x)$.

$$\begin{aligned}
&= \lim_{b\to\infty} \int_{f(3)}^{f(b)} \frac{1}{w^{2/3}}\, dw \\
&= \lim_{b\to\infty} 3w^{\frac{1}{3}}\Big|_{f(3)}^{f(b)} \\
&= \lim_{b\to\infty} 3((f(b))^{\frac{1}{3}} - (f(3))^{\frac{1}{3}}) \\
&= 3(0 - (\frac{\pi}{3})^{1/3}).
\end{aligned}$$