9. [17 points]

a. [4 points] Ivan is studying the series $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ and writes the following argument:

The series is alternating. If we let $a_n = |(-1)^n(1+e^{-n})| = 1+e^{-n}$, then a_n is positive, decreasing, but $\lim_{n\to\infty} a_n = 1$ is not 0. Therefore $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges by the alternating series test.

Ivan's instructor tells Ivan that even though the $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ does diverge, the above argument is incorrect. Explain what's wrong with this argument, and give a correct argument to show that $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges.

Solution: Alternating series test can **only** show convergence. It should not be used to show divergence.
For this series, since $(-1)^n(1+e^{-n})$ oscillates near 1 and -1 as $n \to \infty$, the limit $\lim_{n\to\infty}(-1)^n(1+e^{-n})$ does not exist, in particular not 0. By n-term test, the series $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges.

b. [6 points] Determine whether the following series converge or diverge. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

Solution: Since this series involves factorials, we apply ratio test.

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{(n+1)!}{(2n+3)!}\cdot\frac{(2n+1)!}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{(2n+1)!}{(2n+3)!}\right| \\
&= \lim_{n\to\infty}\left|(n+1)\cdot\frac{1}{(2n+2)(2n+3)}\right| \\
&= 0 < 1.
\end{aligned}$$

Since the limit is < 1, the series $\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!}$ converges by ratio test.

c. [7 points] Determine whether the following series **converges absolutely, converges conditionally, or diverges**. Be sure to fully justify your answer, using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

Solution: We first see if the series converges absolutely by considering

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

To consider this series, we apply integral test. The function $f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing for $x \geq 2$. As for the integral,

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\, dx.$$

Substitute with $w = \ln x$.

$$\begin{aligned} &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{w^2}\, dw \\ &= \lim_{b\to\infty} \left(\frac{-1}{w} \right)\Bigg|_{\ln 2}^{\ln b} \\ &= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}. \end{aligned}$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test. Hence the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges absolutely.