10. [8 points] Determine whether the following series converges or diverges. If it converges, determine if it is absolute or conditional convergence **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n^{1/2}+4)}{3n^{3/2}+2}$$

Circle one: **Absolutely Converges** **Conditionally Converges** **Diverges**

Solution: First, we see that if we ignore the $(-1)^n$, checking leading terms tells us this series behaves like $\frac{1}{n}$, which should diverge. Since we do have the $(-1)^n$, this is an alternating series, and so we should be trying to show this is conditionally convergent.

To apply the alternating series test, we note that the sequence of $a_n = \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ is a decreasing sequence($a_n \geq a_{n+1}$), always positive ($a_n > 0$) , and the limit approaches zero $\lim_{n\to\infty} a_n = 0$. Since the series is alternating, we satisfy the hypothesis of the alternating series test, and so the sequence converges.

To conclude conditionally convergent, we need to show the series:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n^{1/2}+4)}{3n^{3/2}+2} \right| = \sum_{n=1}^{\infty} \frac{(n^{1/2}+4)}{3n^{3/2}+2}$$

diverges. Like we noted above, this functions behaves like $\frac{1}{n}$, so we need to use limit comparison or comparison test with this series to show it diverges.

Solution: Limit Comparison: Let $a_n = \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ and $b_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{(n^{1/2}+4)}{3n^{3/2}+2}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n(n^{1/2}+4)}{3n^{3/2}+2} = \lim_{n\to\infty} \frac{n^{3/2}+4n}{3n^{3/2}+2}.$$

Using L'Hopital, or a dominating functions argument, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}+4n}{3n^{3/2}+2} = \frac{1}{3}$$

Since this limit exists and is non-zero, the limit comparison test tells us that both $\sum_{n=1}^{\infty} \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ and $\sum_{n=1}^{\infty} \dfrac{1}{n}$ will either both converge or both diverge. Since $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges by p-series test with $p = 1$, $\sum_{n=1}^{\infty} \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ diverges by limit comparison test.

Solution: (Direct) Comparison Test: We can also use the direct comparison test. Since we are trying to conclude divergence, we must bound below by a divergent series.We need that $n^{1/2} \leq n^{1/2} + 2$ for all $n \geq 1$ to bound our numerator below. Then we use $3n^{3/2} + 2 \leq 3n^{3/2} + n^{3/2} = 4n^{3/2}$, and so

$$\frac{1}{4n^{3/2}} \leq \frac{1}{3n^{3/2} + 2}$$

for $n \geq 2$, to bound the denominator below. Putting this together, we get

$$\frac{1}{4n} = \frac{n^{1/2}}{4n^{3/2}} \leq \frac{(n^{1/2} + 4)}{3n^{3/2} + 2}.$$

Then, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test, with $p = 1$. Then, by (direct) comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{1/2} + 4)}{3n^{3/2} + 2}$ diverges.