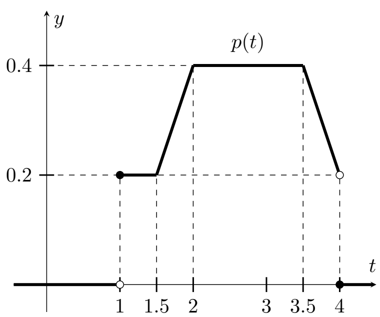
1. [15 points] There are 1000 contestants taking part in the reality show *Endure and Survive*. In the show's first contest, the contestants must complete a timed run of a 500-meter obstacle course. Those who finish within 2 minutes avoid elimination. At the conclusion of this game, the show's director creates a **probability density function** $p(t)$ (shown in the graph below) to model the distribution of the finishing times for the 1000 contestants, where t is given in minutes.



a. [4 points] What was the median finishing time for the 1000 contestants according to the director's model? Include units.

Solution: We find the value M, such that $\int_{-\infty}^{M} p(t)\,dt = 0.5$. First, note $p(t) = 0$, for $t < 1$.
Next, note that $\int_1^2 p(t)\,dt = 1 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.5 = 0.2 + 0.05 = 0.25$. In other words, the area accumulated under $p(t)$ until $t = 2$ is 0.25. Now, note that $\int_2^{3.5} p(t)\,dt = 0.4 \cdot 1.5 = 0.6$. Therefore, the area accumulated under $p(t)$ until $t = 3.5$ is $0.25 + 0.6 = 0.85$. This tells us that $2 < M < 3.5$. Given this, we need

$$\begin{aligned} 0.5 &= \int_1^M p(t)\,dt \\ &= \int_1^2 p(t)\,dt + \int_2^M p(t)\,dt \\ &= 0.25 + 0.4(M-2) \\ 0.25 &= 0.4(M-2) \\ M &= 2 + \frac{0.25}{0.4} = 2 + \frac{5}{8} = \frac{21}{8} = 2.625. \end{aligned}$$

Therefore, the median finishing time for the 1000 contestants, according to the director's model, was 2.625 minutes.

Answer: 2.625 minutes

b. [2 points] According to the director's model, how many of the 1000 contestants were eliminated in this game?

Solution: According to the director's model, the fraction of contestants eliminated is given by

$$\int_2^{\infty} p(t)\,dt = 1 - \int_1^2 p(t)\,dt = 1 - 0.25 = 0.75.$$

Therefore, $0.75 \cdot 1000 = 750$ contestants were eliminated in this game.

Answer: 750 contestants

1. (continued) For the next segment of the show, each remaining contestant goes through the obstacle course again. This time, their goal is to travel as much distance within the obstacle course as they can in one minute. The **probability density function**, $r(x)$, for the amount of distance x (measured in hundreds of meters) the remaining contestants travel within the obstacle course in one minute is given by

$$r(x) = \begin{cases} \frac{1}{30}(3x^2 - 3) & \text{for } 1 < x < 3, \\ a(x-5)^2 & \text{for } 3 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

c. [6 points] Find the value of a so that $r(x)$ is a probability density function.

Solution: For $r(x)$ to be a probability density function, it must satisfy $\int_{-\infty}^{\infty} r(x)\,dx = 1$. Since $r(x)$ is only nonzero on the interval $1 < x < 5$, we need it to satisfy,

$$\int_1^3 \frac{1}{30}(3x^2 - 3)\,dx + \int_3^5 a(x-5)^2\,dx = 1.$$

Solving, we get

$$\frac{1}{30}(x^3 - 3x)\Big|_1^3 + \frac{a(x-5)^3}{3}\Big|_3^5 = 1$$
$$\frac{1}{30}(27 - 9 - (1-3)) + a(0 - (-\frac{8}{3})) = 1$$
$$\frac{2}{3} + \frac{8}{3}a = 1 \implies a = \frac{1}{8}.$$

Answer: $a = \underline{\mathbf{\frac{1}{8}}}$

d. [3 points] Below, circle the **one** best interpretation of the equation

$$r(2) = 0.3.$$

- 30% of the remaining contestants travel exactly 200 meters within the obstacle course in one minute.
- Approximately 30% of the remaining contestants travel exactly 200 meters within the obstacle course in one minute.
- 60% of the remaining contestants travel at most 200 meters within the obstacle course in one minute.
- Approximately 1.5% of the remaining contestants travel between 200 meters and 205 meters within the obstacle course in one minute.
- Approximately 1.5% of the remaining contestants travel between 195 meters and 205 meters within the obstacle course in one minute.
- Approximately 2% of the remaining contestants travel exactly 300 meters within the obstacle course in one minute.