7. [16 points] A treasure hunter has spotted a large exotic rock at the bottom of a deep pit. The vertical distance from the top of the pit to the top of the rock is 15 meters. To retrieve the rock, the treasure hunter attaches a 15 meter rope to the top of the rock and lifts it out of the pit. The rope used has mass 2 kg per meter. Below, **do not simplify your final answers or evaluate any integrals**. As a reminder, the acceleration due to gravity is g, where $g = 9.8\,\mathrm{m/s^2}$.

a. [8 points] If the rock has mass 4 kg, write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the rock and the attached rope 10 meters up from the bottom of the pit.

Hint: Once rope has been raised to the top of the pit, the treasure hunter no longer needs to lift it.

Solution: After lifting the rock and the attached rope a distance of h meters, the length of the attached rope yet to be lifted is $15 - h$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$4 + 2 \cdot (15 - h) \text{ kg.}$$

So, the work done to lift the rock and the attached rope a short distance Δh meters at this moment is,

$$(4 + 2 \cdot (15 - h)) \cdot 9.8 \cdot \Delta h \text{ Joules.}$$

Therefore, the amount of work done by the treasure hunter to lift the rock and the attached rope 10 meters up from the bottom of the pit is given by,

$$\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8\, dh \text{ Joules.}$$

Answer: $\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8\, dh$

b. [8 points] After the rock has been lifted 10 meters off the bottom of the pit, the rock starts to crumble, losing 0.1 kg of mass per second. The treasure hunter resumes lifting the rock at a constant speed of 0.5 meters per second. Write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit.

The hint from part ***a.*** *still applies.*

Solution: Since the rock loses 0.1 kg of mass per second, and the treasure hunter lifts at a constant speed of 0.5 meters per second, we have that the rock loses mass at a rate of

$$\frac{0.1\frac{\text{kg}}{\text{s}}}{0.5\frac{\text{m}}{\text{s}}} = 0.2\frac{\text{kg}}{\text{m}}.$$

Now, after lifting the rock and the attached rope a further distance of x meters, the length of the attached rope yet to be lifted is $5 - x$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$(4 - 0.2x) + 2 \cdot (5 - x) \text{ kg}.$$

So, the work done to lift the rock and the attached rope a short distance Δx meters at this moment is,

$$((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \cdot \Delta x \text{ Joules}.$$

Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit is given by,

$$\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx \text{ Joules}.$$

Alternatively, we can compute quantities after we have lifted the rock and the attached rope for a duration of t seconds. The mass of the crumbling rock is $4 - 0.1t$ kg. Now, the attached rope here starts of with a mass of $2 \cdot 5 = 10$ kg, and every second 0.5 meters of it is being retracted/lifted. So, the mass of the attached rope is given by $10 - 2(0.5 \cdot t) = 10 - t$ kg. The combined mass of the rock and the attached rope is then

$$(4 - 0.1t) + (10 - t) \text{ kg}.$$

Now, in the next short period of Δt seconds, the system is lifted $0.5\Delta t$ meters. So, the work done to lift the rock and the attached rope in this short period is

$$((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5\Delta t \text{ Joules}.$$

Now, at a speed of 0.5 meters per second, it takes 10 seconds to lift the system the remaining 5 meters to the top. Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) in this process is given by,

$$\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt \text{ Joules}.$$

Answer: $\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx$ **or** $\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt$