8. [12 points] The parts of this problem are unrelated to each other.

a. [5 points] Compute the following limit. Fully justify your answer including using proper notation.

$$\lim_{x \to \infty} 2x \ln \left(1 + \frac{5}{x}\right)$$

Solution: We note that $\lim_{x \to \infty} 2x = \infty$, and $\lim_{x \to \infty} \ln \left(1 + \frac{5}{x}\right) = 0$. So, the limit is in the $\infty \cdot 0$ form, allowing us to attempt to use L'Hôpital's rule. So, we first express the limit as a fraction, and proceed as shown below.

$$\lim_{x \to \infty} 2x \ln \left(1 + \frac{5}{x}\right) = 2 \lim_{x \to \infty} \ln \left(1 + \frac{5}{x}\right) \cdot \frac{1}{x}$$

L'H \( \frac{0}{\infty} \)

$$= \lim_{x \to \infty} \frac{1}{x + \frac{5}{x}} \cdot \frac{-5}{x^2}$$

$$= 2 \lim_{x \to \infty} \frac{1}{\frac{1}{x} + \frac{5}{x^2}}$$

$$= 10.$$ 

Answer: $\lim_{x \to \infty} 2x \ln \left(1 + \frac{5}{x}\right) = 10$

b. [7 points] Compute the value of the following improper integral if it converges. If it does not converge, use a direct computation of the integral to show its divergence. Circle your final answer choice. Show your full computation, and use proper notation.

$$\int_{0}^{3} \frac{1}{(x-3)^2} \, dx$$

Circle one: 

Converges to: 

Diverges

Solution: We note that $\frac{1}{(x-3)^2}$ has a vertical asymptote at $x = 3$. Therefore, we express the given (improper) integral as follows, and proceed with direct computation.

$$\int_{0}^{3} \frac{1}{(x-3)^2} \, dx = \lim_{b \to 3} \int_{0}^{b} \frac{1}{(x-3)^2} \, dx$$

$$= \lim_{b \to 3} - \frac{1}{x-3} \bigg|_{0}^{b}$$

$$= \lim_{b \to 3} - \frac{1}{b-3} - \frac{1}{3}$$

$$= +\infty \text{ (or DNE).}$$

Therefore, by direct computation, $\int_{0}^{3} \frac{1}{(x-3)^2} \, dx$ diverges.