8. [12 points] The parts of this problem are unrelated to each other.

a. [5 points] Compute the following limit. Fully justify your answer including using proper notation.

$$\lim_{x\to\infty} 2x \ln\left(1+\frac{5}{x}\right)$$

Solution: We note that $\lim\limits_{x\to\infty} 2x = \infty$, and $\lim\limits_{x\to\infty} \ln\left(1+\frac{5}{x}\right) = 0$. So, the limit is in the "$\infty \cdot 0$" form, allowing us to attempt to use L'Hôpital's rule. So, we first express the limit as a fraction, and proceed as shown below.

$$\begin{aligned}\lim_{x\to\infty} 2x\ln\left(1+\frac{5}{x}\right) &= 2\lim_{x\to\infty}\frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}}\\ &\overset{\text{L'H}}{=} 2\lim_{x\to\infty}\frac{\frac{1}{1+\frac{5}{x}}\cdot\frac{-5}{x^2}}{\frac{-1}{x^2}}\\ &= 2\lim_{x\to\infty}\frac{1}{1+\frac{5}{x}}\cdot 5\\ &= 10.\end{aligned}$$

Answer: $\lim\limits_{x\to\infty} 2x\ln\left(1+\frac{5}{x}\right) =$ __**10**__

b. [7 points] Compute the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. **Circle your final answer choice.** Show your full computation, and use proper notation.

$$\int_0^3 \frac{1}{(x-3)^2}\,dx$$

Circle one: **Converges to:** ______________ $\boxed{\textbf{Diverges}}$

Solution: We note that $\frac{1}{(x-3)^2}$ has a vertical asymptote at $x = 3$. Therefore, we express the given (improper) integral as follows, and proceed with direct computation.

$$\begin{aligned}\int_0^3 \frac{1}{(x-3)^2}\,dx &= \lim_{b\to 3^-}\int_0^b \frac{1}{(x-3)^2}\,dx\\ &= \lim_{b\to 3^-} -\frac{1}{x-3}\Big|_0^b\\ &= \lim_{b\to 3^-} -\frac{1}{b-3} - \frac{1}{3}\\ &= +\infty \text{ (or DNE).}\end{aligned}$$

Therefore, by direct computation, $\int_0^3 \frac{1}{(x-3)^2}\,dx$ diverges.