2. [12 points] Joe and Paula are at the same national park, hiking through the forest. They arrive at the bottom of a cliff and challenge each other to bring their hiking gear to the top of the cliff, which is 25 meters above the bottom. Each of them has a different idea of how to accomplish this.
You may assume that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

a. [6 points] Joe plans to climb to the top of the cliff while carrying his water bottle. Before Joe starts climbing, the combined mass of Joe and his water bottle is 64 kilograms. However, a rock punctures the bottle as soon as Joe starts climbing, so water leaks out at a constant rate of 0.03 kilograms per second. Joe climbs the cliff at a constant rate of 0.25 meters per second.

(i) Let $M(h)$ be the combined mass of Joe and his water bottle, in kilograms (kg), when Joe is h meters above the ground. Write an expression for $M(h)$.

Solution: Water leaks out at a rate of 0.03 kg/s, and Joe climbs the cliff at a rate of 0.25 m/s, so $0.03/0.25 = 0.12$ kg of water leak out for every meter that Joe climbs. So the combined mass of Joe and the water bottle, when Joe is h meters above the ground, is $64 - 0.12h$ kg.

Answer: $M(h) =$ $64 - 0.12h$

(ii) Write an integral representing the total amount of work, in Joules (J), that it takes for Joe to move himself and the water bottle to the top of the cliff. Your answer should not involve the letter M. **Do not evaluate your integral.**

Solution: Below are two possible ways to solve this problem:

Solution 1 (Work in terms of height)**:** Recall that (work) = (force)(distance). The force equals $9.8(64 - 0.12h)$ N, as this is the mass $M(h) = 64 - 0.12h$ kg from part (i) times the acceleration 9.8 m/s^2. When Joe is at a height of h meters, the amount of work it takes for him to carry the water bottle an additional Δh meters is approximately $9.8(64 - 0.12h)\,\Delta h$ J. Therefore the total amount of work it takes for him to carry the water bottle 25 meters is

$$\int_0^{25} 9.8(64 - 0.12h)\,dh \text{ J}.$$

Solution 2 (Work in terms of time)**:** Alternatively, we can compute the work in terms of time without using our answer from part (i). The combined mass of Joe and the water bottle is 64 kilograms when Joe starts climbing the cliff, and this mass decreases at a rate of 0.03 kg/s due to the water bottle leaking. So, the combined mass of Joe and the water bottle t seconds after Joe starts climbing is $64 - 0.03t$ kg.

Recall that (work) = (force)(distance). The force equals $9.8(64 - 0.03t)$ N, as this is the mass $64 - 0.03t$ kg times the acceleration 9.8 m/s^2. To find the distance, we note that Joe climbs the cliff at a rate of 0.25 m/s. So, when he has been climbing the cliff for t seconds, he will climb an additional $0.25\,\Delta t$ m over the next Δt seconds. Thus the amount of work it takes for him to carry the water bottle an additional Δt seconds is approximately $9.8(0.25)(64 - 0.03t)\,\Delta t$ J. Since the cliff is 25 m tall, and Joe climbs the cliff at a rate of 0.25 m/s, then it takes $25/0.25 = 100$ seconds for Joe to climb the cliff. Therefore the total amount of work it takes for him to carry the water bottle 25 meters is

$$\int_0^{100} 9.8(0.25)(64 - 0.03t)\,dt \text{ J}.$$

Answer: $\int_0^{25} 9.8(64 - 0.12h)\,dh$ **OR** $\int_0^{100} 9.8(0.25)(64 - 0.03t)\,dt$

b. [6 points] Paula ties a rope to her 3-kilogram backpack, walks to the top of the cliff, and then uses the rope to pull her backpack to the top. The rope has a mass of 0.1 kilograms per meter. Write an integral representing the total amount of work, in Joules (J), that it takes for Paula to pull her backpack and the attached rope to the top of the cliff.
Do not evaluate your integral.

Solution: Below are two possible ways to solve this problem:

Solution 1 (Slicing the height)**:** When Paula has already lifted her backpack a total of h meters, the remaining rope she must pull up has a length of $25 - h$ meters, and thus a mass of $0.1(25 - h)$ kilograms. Together with the backpack, the amount of mass that remains for her to lift up is $3 + 0.1(25 - h)$ kg. At this point, the amount of work it takes to lift the rope and the backpack an additional Δh meters is approximately $9.8(3 + 0.1(25 - h))\Delta h$ J. Therefore the total amount of work it takes for her to pull the backpack up 25 meters is

$$\int_0^{25} 9.8(3 + 0.1(25 - h))\, dh \text{ J.}$$

Solution 2 (Slicing the rope)**:** The work required to lift only the backpack up to the top of the cliff is $9.8(3)(25) = 9.8(75)$ J. Consider a segment of rope of length $\Delta\ell$ meters which is ℓ meters from the top of the cliff. The mass of this segment is $0.1\,\Delta\ell$, and the amount of work required to lift this segment to the top of the cliff is $9.8(0.1)\ell\,\Delta\ell$. Thus the amount of work required to lift the entire rope to the top of the cliff is $\int_0^{25} 9.8(0.1)\ell\, d\ell$. Therefore the total amount of work required to lift the backpack and the rope to the top of the cliff is

$$9.8(75) + \int_0^{25} 9.8(0.1)\ell\, d\ell \text{ J.}$$

Answer: $\int_0^{25} 9.8(3 + 0.1(25 - h))\, dh$ **OR** $9.8(75) + \int_0^{25} 9.8(0.1)\ell\, d\ell$