4. [11 points] Zach is playing the retro video game *Plaque-Man* all day to get a new personal high score. Zach starts playing the game with 0 points. Over the course of each hour, Zach scores an additional 2500 points. At the **beginning** of every hour, Zach trades 20% of his points to buy extra time. For $n \geq 1$, let H_n be Zach's score at the **end** of the nth hour of playing the game.
For example, $H_1 = 2500$.

a. [4 points] Write expressions for H_2 and H_3. Your answers should not involve the letter H. You do not need to simplify your expressions.

Solution: By definition, H_n is Zach's score at the end of the nth hour. At the beginning of the $(n+1)$th hour, Zach trades 20% of his points to buy extra time, so he has $0.8H_n$ points at the beginning of the $(n+1)$th hour. Over the course of the $(n+1)$th hour, Zach gets 2500 more points, so at the end of the $(n+1)$th hour, Zach's score is $H_{n+1} = 0.8H_n + 2500$. Since $H_1 = 2500$, we can use this recursive formula to write expressions for H_2 and H_3, shown below.

$H_2 =$ $2500(0.8) + 2500$

$H_3 =$ $2500(0.8)^2 + 2500(0.8) + 2500$

b. [4 points] Write a **closed-form** expression for H_n. *Closed-form* means your answer should not include ellipses $(\ldots)$ or sigma notation (Σ), and should not be recursive. You do not need to simplify your closed-form expression.

Solution: Continuing the pattern above from our expressions of H_1, H_2, and H_3, we find that

$$H_n = 2500(0.8)^{n-1} + \cdots + 2500(0.8)^2 + 2500(0.8) + 2500.$$

This is a finite geometric series. Recall the formula for the sum of a finite geometric series:

$$a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1-x^n)}{1-x}, \qquad x \neq 1.$$

In the context of this problem, $a = 2500$ and $x = 0.8$, so we obtain our answer.

Answer: $H_n = \dfrac{2500(1-0.8^n)}{1-0.8}$

c. [3 points] Find Zach's eventual score if he keeps playing *Plaque-Man* indefinitely. You do not need to simplify your numerical answer.

Solution: Zach's eventual score is found by taking a limit of the sequence H_n as $n \to \infty$:

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} \frac{2500(1-0.8^n)}{1-0.8} = \frac{2500}{1-0.8} = 12500.$$

Therefore Zach's eventual score will be 12500.

Answer: 12500