5. [14 points]

a. [7 points] Determine whether the following improper integral is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. You do not need to compute the value of the integral if it is convergent. Circle your final answer choice.

$$\int_1^\infty \frac{4+\sin(x)}{x^3+2}\,dx$$

Circle one: **Convergent** **Divergent**

Solution: Note that 4 is larger than $\sin(x)$ for all x, and also x^3 dominates 2 for large x. Our intuition tells us that, for sufficiently large x,

$$\frac{4+\sin(x)}{x^3+2} \approx \frac{4}{x^3}.$$

Since $\int_1^\infty \frac{4}{x^3}\,dx$ converges, we expect the original improper integral to converge. We now need to make this reasoning rigorous.

Note that $-1 \le \sin(x) \le 1$ and $x^3+2 \ge x^3$ for all $x \ge 1$. This shows that

$$0 \le \frac{4+\sin(x)}{x^3+2} \le \frac{4+1}{x^3+2} \le \frac{4+1}{x^3} = \frac{5}{x^3} \qquad \text{for all } x \ge 1.$$

By the ***p*-test** ($p = 3$)**,** the integral $\int_1^\infty \frac{5}{x^3}\,dx$ is **convergent.**

Therefore, by the **comparison test,** the integral $\int_1^\infty \frac{4+\sin(x)}{x^3+2}\,dx$ is **convergent.**

b. [7 points] Let $0 < p < 1$ be a real number, and consider the improper integral

$$\int_1^3 \frac{1}{t(\ln(t))^p}\,dt.$$

The integral above converges; to show this, **compute** its value. Your answer may involve p. Be sure to show your full computation, and be sure to use **proper notation.** Remember: $0 < p < 1$.

Solution: To evaluate this integral, we use proper notation and the substitution $w = \ln(t)$, so that $dw = \frac{1}{t}\,dt$, and so

$$\int_1^3 \frac{1}{t(\ln(t))^p}\,dt = \lim_{a\to 1^+}\int_a^3 \frac{1}{t(\ln(t))^p}\,dt = \lim_{a\to 1^+}\int_{\ln(a)}^{\ln(3)} \frac{1}{w^p}\,dw$$

$$= \lim_{a\to 1^+}\left(\frac{w^{1-p}}{1-p}\Bigg|_{\ln(a)}^{\ln(3)}\right)$$

$$= \lim_{a\to 1^+}\left(\frac{\ln(3)^{1-p}}{1-p} - \frac{\ln(a)^{1-p}}{1-p}\right) = \frac{\ln(3)^{1-p}}{1-p}.$$

(Note that $\ln(1)^{1-p} = 0$, as opposed to being undefined, since $0 < p < 1$ means that $1-p > 0$.)

Answer: $\int_1^3 \frac{1}{t(\ln(t))^p}\,dt =$ $\frac{\ln(3)^{1-p}}{1-p}$