6. [10 points] Liban is writing songs using a new style of music which he calls "new-age jazz." The longer that he spends writing a particular song, the better it turns out.

Let $Q(t)$ be the **cumulative distribution function** (cdf) for t, the number of days that it takes for Liban to write a particular song. The formula for $Q(t)$ is shown to the right, where $c > 0$ is a constant.

$$Q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{4}t^2 & 0 \le t \le 2, \\ 2c - c\,e^{2-t} & t > 2. \end{cases}$$

You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [3 points] Write a piecewise-defined formula for $q(t)$, the **probability density function** (pdf) corresponding to $Q(t)$. Your answer may involve c, but it should not involve the letter Q.

Solution: We know that $Q(t)$ and $q(t)$ are related by the formula $Q'(t) = q(t)$. So, the formula for $q(t)$ is found by taking the derivative of each part of $Q(t)$.

$$q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{2}t & 0 \le t \le 2, \\ c\,e^{2-t} & t > 2. \end{cases}$$

b. [3 points] Write an expression involving one or more integrals that represents the **mean** number of days that it takes for Liban to write a particular song. Your answer may involve c, but it should not involve the letters Q or q. **Do not evaluate your integral(s).**

Solution: The formula for the mean is given by $\int_{-\infty}^{\infty} tq(t)\,dt$. Using our answer to part (a):

$$\int_{-\infty}^{\infty} tq(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt = \int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt.$$

Answer: $\int_0^2 \frac{c}{2}t^2\,dt + \int_2^{\infty} c\,te^{2-t}\,dt$

c. [2 points] Use the fact that $Q(t)$ is a cumulative distribution function to find the value of c.

Solution: Since $Q(t)$ is a cumulative distribution function, we must have $\lim_{t\to\infty} Q(t) = 1$. Using the formula for $Q(t)$ for $t > 2$,

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} (2c - c\,e^{2-t}) = 2c.$$

Therefore $2c = 1$, so that $c = \frac{1}{2}$.

Answer: $c = \frac{1}{2}$

d. [2 points] Circle the **one** correct answer below that completes the following sentence:

"The quantity $Q(5)$ represents...

(i) *...the probability that it takes exactly 5 days for Liban to write a song."*

(ii) *...the probability that it takes more than 5 days for Liban to write a song."*

$\boxed{\text{(iii)}}$ *...the probability that it takes 5 days or less for Liban to write a song."*

(iv) *...the approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song."*

(v) NONE OF THESE

Solution: We know that $Q(5)$ and $q(5)$ are related by the formula $Q(5) = \int_{-\infty}^{5} q(t)\,dt$, and this integral represents the probability that it takes 5 days or less for Liban to write a song. To explain why the other choices are incorrect:

In general, the probability that it takes Liban between a and b days to write a song is the quantity

$$Q(b) - Q(a) = \int_a^b q(t)\,dt.$$

Thus the probability that it takes exactly 5 days for Liban to write a song must be

$$Q(5) - Q(5) = \int_5^5 q(t)\,dt = 0.$$

But $Q(5) \neq 0$ since we know $Q(t)$ is a nondecreasing function (as it is a cdf), and so we have $Q(5) \geq Q(2) = 0.5 > 0$, using our value of c from part (c). So (i) is incorrect.

The probability that it takes more than 5 days for Liban to write a song is $1 - Q(5) = \int_5^{\infty} q(t)\,dt$. We have $Q(2) = 0.5$. The formula for $Q(t)$ shows that it is strictly increasing, so $Q(5) > 0.5$, and thus $1 - Q(5) < 0.5$. This means $Q(5) \neq 1 - Q(5)$, so (ii) is incorrect.

The approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song is a standard interpretation of the quantity $q(5)$, which describes a pdf, not a cdf. So (iv) is also incorrect.