7. [7 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use.
Circle your final answer choice.

$$\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$$

Circle one: **Convergent** **Divergent**

Solution: We claim that the series is convergent. To show this, we use the ratio test. Let $a_n = \frac{4^n \cdot n^2}{n!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!}\right|}{\left|\frac{4^n \cdot n^2}{n!}\right|} = \lim_{n\to\infty} \left|\frac{4^{n+1}\cdot(n+1)^2}{(n+1)!} \cdot \frac{n!}{4^n \cdot n^2}\right| = \lim_{n\to\infty} \frac{4(n+1)}{n^2} = 0 < 1.$$

Therefore, by the **ratio test,** the series $\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$ is **convergent.**

8. [6 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1}$$

Solution: We first note that this limit is of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)}.$$

But now this new limit is again of the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule a second time:

$$\lim_{x\to 0} \frac{2e^{2x} - 2(x+1)}{-\sin(x)} \overset{L'H\ \frac{0}{0}}{=} \lim_{x\to 0} \frac{4e^{2x} - 2}{-\cos(x)} = \frac{4-2}{-1} = -2.$$

Therefore the original limit equals -2.

Answer: -2

©*2024 Univ of Michigan Dept of Mathematics*
Creative Commons BY-NC-SA 4.0 International License