9. [12 points] Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$$

Circle one: **Absolutely Convergent** **Conditionally Convergent** **Divergent**

Solution: Observe that the given series is of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n, \qquad \text{where} \qquad a_n = \frac{n}{n^2 + n^{1/2}}.$$

The sequence a_n satisfies $0 < a_{n+1} < a_n$ for all $n \geq 1$, and also $\lim\limits_{n\to\infty} a_n = 0$.

Therefore, by the **alternating series test,** the series $\sum\limits_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$ is **convergent.**

On the other hand, we claim that the original series is not absolutely convergent. To do this, we will show that the series

$$\sum_{n=1}^{\infty} \left| \frac{n \cdot (-1)^n}{n^2 + n^{1/2}} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$$

is divergent. There are two possible ways to do this:

Solution 1 (Direct comparison test)**:** Note that $n^{1/2} \leq n^2$ for $n \geq 1$, so that

$$\frac{n}{n^2 + n^{1/2}} \geq \frac{n}{n^2 + n^2} = \frac{1}{2n} \geq 0 \qquad \text{for all } n \geq 1.$$

By the ***p*-series test** $(p = 1)$**,** the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ is **divergent.**

So, by the **direct comparison test,** the series $\sum\limits_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$ is **divergent.**

Solution 2 (Limit comparison test)**:** Consider the following limit computation:

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+n^{1/2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2 + n^{1/2}} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n^{3/2}}} = 1 > 0.$$

Note that this limit exists and is positive.

By the ***p*-series test** $(p = 1)$**,** the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is **divergent.**

So, by the **limit comparison test,** the series $\sum\limits_{n=1}^{\infty} \frac{n}{n^2 + n^{1/2}}$ is **divergent.**

Therefore, the original series $\sum\limits_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$ is **conditionally convergent.**