6. [12 points] Kalani and his friends are in Honolulu, Hawaii, enjoying their spring break. One day, they decide to try a new virtual surfing game. Let P(t) denote the cumulative distribution function (cdf) representing the probability that a first-time player takes t minutes or less to complete the game. According to the developer's data, the formula of P(t) is given by

$$P(t) = \begin{cases} 0, & t \le 0, \\ \frac{a}{9}t^2, & 0 < t \le 3, \\ \frac{b}{2} - ae^{3-t}, & t > 3. \end{cases}$$

where a > 0 and b > 0 are constants.

a. [4 points] The function P(t) is a **continuous** cumulative distribution function (cdf). Find the values of a and b.

Solution: Since P(t) is a cumulative distribution function (cdf), it must satisfy $\lim_{t\to\infty} P(t) = 1$. That is,

$$\lim_{t \to \infty} \left(\frac{b}{2} - ae^{3-t}\right) = \frac{b}{2} - 0 = \frac{b}{2} = 1$$

Therefore, b = 2. Since P(t) is a continuous function, the following must hold:

$$\frac{a}{9} \cdot 9 = \frac{2}{2} - ae^{3-3} \implies a = 1 - a$$

Hence, $a = \frac{1}{2}$.

Answer:
$$a = \underline{\qquad \qquad \frac{1}{2} \qquad \qquad \text{and } b = \underline{\qquad \qquad 2}$$

b. [2 points] Write an expression for the probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game. Your answer may include the letters a and b, but it should not involve the letter P. Your answer should **not** include integrals.

Solution: The probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game is given by

$$P(7) - P(1) = \left(\frac{b}{2} - ae^{3-7}\right) - \frac{a(1)^2}{9} = \left(\frac{b}{2} - ae^{-4}\right) - \frac{a}{9} = \left(1 - \frac{e^{-4}}{2}\right) - \frac{1}{18}.$$
Answer:
$$\frac{17}{18} - \frac{e^{-4}}{2}$$