6. [12 points] Kalani and his friends are in Honolulu, Hawaii, enjoying their spring break. One day, they decide to try a new virtual surfing game. Let $P(t)$ denote the cumulative distribution function (cdf) representing the probability that a first-time player takes t minutes or less to complete the game. According to the developer's data, the formula of $P(t)$ is given by

$$P(t) = \begin{cases} 0, & t \le 0, \\ \frac{a}{9}t^2, & 0 < t \le 3, \\ \frac{b}{2} - ae^{3-t}, & t > 3. \end{cases}$$

where $a > 0$ and $b > 0$ are constants.

a. [4 points] The function $P(t)$ is a **continuous** cumulative distribution function (cdf). Find the values of a and b.

Solution: Since $P(t)$ is a cumulative distribution function (cdf), it must satisfy $\lim_{t\to\infty} P(t) = 1$. That is,

$$\lim_{t\to\infty}\left(\frac{b}{2} - ae^{3-t}\right) = \frac{b}{2} - 0 = \frac{b}{2} = 1$$

Therefore, $b = 2$. Since $P(t)$ is a continuous function, the following must hold:

$$\frac{a}{9}\cdot 9 = \frac{2}{2} - ae^{3-3} \implies a = 1 - a$$

Hence, $a = \frac{1}{2}$.

Answer: $a =$ $\frac{1}{2}$ and $b =$ 2

b. [2 points] Write an expression for the probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game. Your answer may include the letters a and b, but it should not involve the letter P. Your answer should **not** include integrals.

Solution: The probability that a first-time player takes at least 1 minute and at most 7 minutes to complete the game is given by

$$P(7) - P(1) = \left(\frac{b}{2} - ae^{3-7}\right) - \frac{a(1)^2}{9} = \left(\frac{b}{2} - ae^{-4}\right) - \frac{a}{9} = \left(1 - \frac{e^{-4}}{2}\right) - \frac{1}{18}.$$

Answer: $\frac{17}{18} - \frac{e^{-4}}{2}$

c. [3 points] Write a piecewise-defined formula for $p(t)$, the probability density function (pdf) corresponding to $P(t)$. Your answer may include the letters a and b, but it should not include the letter P.

Solution: Since $P(t)$ is an anti-derivative of $p(t)$, we have $p(t) = P'(t)$. Therefore,

$$p(t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{t}{9}, & 0 < t \leq 3, \\ \dfrac{e^{3-t}}{2}, & t > 3. \end{cases}$$

d. [3 points] Write an expression involving one or more integrals that represents the mean time (in minutes) it takes for a first-time player to complete the virtual surfing game. Your answer may include the letters a and b, but it should not involve the letters P or p. **Do not evaluate your integral(s).**

Solution: The mean time is given by

$$\int_0^3 \frac{t^2}{9}\,\mathrm{d}t + \int_3^\infty \frac{te^{3-t}}{2}\,\mathrm{d}t$$

Answer: $\displaystyle\int_0^3 \frac{t^2}{9}\,\mathbf{d}t + \int_3^\infty \frac{te^{3-t}}{2}\,\mathbf{d}t$