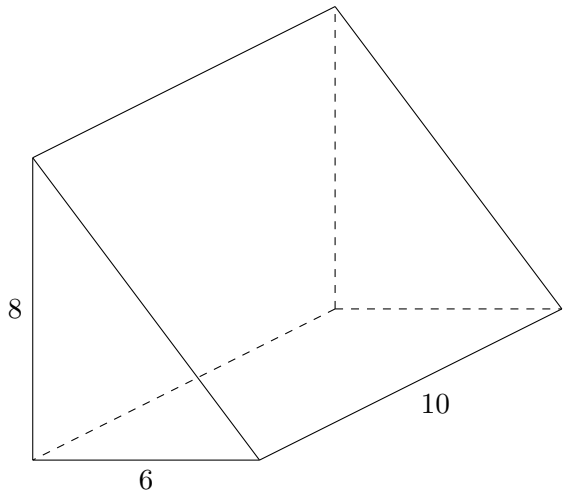
8. [12 points]

Caroline the water engineer is helping to clean out a large tank which is partially filled with sludge. The tank is in the shape of a right triangular prism, with height 8 meters, width 6 meters, and length 10 meters, as depicted to the right. The sludge fills the tank to a depth of 5 meters, so that the top of the sludge is 3 meters below the top of the tank. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [2 points] The width of a horizontal slice of sludge at the bottom of the tank is 6 meters. Find an expression for the width, in meters, of a horizontal slice which is h meters above the bottom of the tank.

Solution: Denote by w the width of a horizontal slice of sludge located h meters above the bottom of the tank. We use similar triangles to determine the relationship between w and h:

$$\frac{w}{6} = \frac{8-h}{8} \implies w = \frac{6(8-h)}{8} = \frac{3(8-h)}{4}.$$

Answer: $\dfrac{3(8-h)}{4}$

b. [5 points] The density, in kilograms per cubic meter, of the sludge at a height h meters above the bottom of the tank is given by the function $p(h)$. Find an expression which approximates the **weight** of a horizontal slice of the sludge which is h meters above the bottom of the tank, and which has small thickness Δh. Your expression should not contain any integrals. **Include units**.

Solution: The approximate volume ΔV of a horizontal slice of sludge at a height h meters above the bottom of the tank, with a small thickness Δh, is given by

$$\Delta V \approx 10 w \Delta h = 10\left(\frac{3(8-h)}{4}\right)\Delta h = \frac{15(8-h)}{2}\Delta h.$$

Thus, the approximate mass Δm of this slice is

$$\Delta m \approx \Delta V \times p(h) = \frac{15(8-h)p(h)}{2}\Delta h$$

And, therefore, the approximate weight ΔF of this slice is

$$\Delta F \approx \Delta m \times g = \frac{15g(8-h)p(h)}{2}\Delta h$$

Answer: $\dfrac{15g(8-h)p(h)}{2}\Delta h$ **Units:** Newton (N) or kgms^{-2}

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c. [5 points] Recall that the sludge fills the tank to a depth of 5 meters, so that the top of the sludge is 3 meters below the top of the tank. Write an expression involving one or more integrals that gives the total amount of work needed to pump all the sludge to the top of the tank. Do not evaluate your integral(s). **Include units.**

Solution: Denote by ΔW the approximate work required to pump a horizontal slice of sludge at a height h meters above the bottom of the tank, with a small thickness Δh. This slice is moved a distance of $(8-h)$ meters to be pumped out. We have

$$\Delta W \approx \Delta F \times (8-h) = \frac{15g(8-h)^2 p(h)}{2} \Delta h$$

Since the sludge fills the tank to a depth of 5 meters, the total work required to pump all the sludge to the top of the tank is given by

$$\int_0^5 \frac{15g(8-h)^2 p(h)}{2} \, \mathrm{d}h$$

Answer: $\displaystyle\int_0^5 \frac{15g(8-h)^2 p(h)}{2} \, \mathrm{d}h$ **Units:** Joules (J) or $\mathrm{kgm^2 s^{-2}}$

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