1. [11 points] Suppose $p(t)$ is the **probability density function** for a variable t.

$$p(t) = \begin{cases} 0 & t \leq 0 \\ at^2 & 0 < t \leq \frac{1}{2} \\ t & \frac{1}{2} < t \leq 1 \\ 0 & t > 1 \end{cases}$$

a. [3 points] Find the value of a such that $p(t)$ is a probability density function.

Solution: Since $p(t)$ is a pdf, we know that $\int_{-\infty}^{\infty} p(t)dt = 1$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} p(t)dt \\ &= \int_0^{\frac{1}{2}} at^2 \, dt + \int_{\frac{1}{2}}^1 t \, dt \\ &= \frac{at^3}{3}\Big|_0^{\frac{1}{2}} + \frac{t^2}{2}\Big|_{\frac{1}{2}}^1 \\ &= \frac{a}{24} + \frac{1}{2} - \frac{1}{8} \end{aligned}$$

Solve for a to get $a = \frac{5}{8}(24) = 15$.

Answer: $a =$ 15

b. [3 points] Write an expression involving one or more integrals for the mean of the distribution. **Do not evaluate any integrals in your answer**. Your final answer should not include the letters p or P, but may include the letter a.

Solution: The mean of the distribution is given by

$$\int_0^{1/2} at^3 \, dt + \int_{1/2}^1 t^2 \, dt = \int_0^{1/2} 15t^3 \, dt + \int_{1/2}^1 t^2 \, dt$$

Answer: $\int_0^{1/2} 15t^3 \, dt + \int_{1/2}^1 t^2 \, dt$

c. [5 points] Write a formula for the corresponding cumulative distribution function, $P(t)$. Your formula should not contain any integral signs, but may include the letter a.

Solution:
The function $P(t)$ must be a continuous antiderivative of $p(t)$ which satisfies $\lim_{t\to-\infty} P(t) = 0$ and $\lim_{t\to\infty} P(t) = 1$. We have:

$$P(t) = \int_{-\infty}^{t} p(x)\ dx = \begin{cases} 0 & t \le 0 \\ \frac{a}{3}t^3 & 0 < t \le \frac{1}{2} \\ \frac{a}{24} + \frac{t^2}{2} - \frac{1}{8} & \frac{1}{2} < t \le 1 \\ 1 & t > 1 \end{cases}$$

Plugging in $a = 15$,

$$P(t) = \int_{-\infty}^{t} p(x)\ dx = \begin{cases} 0 & t \le 0 \\ 5t^3 & 0 < t \le \frac{1}{2} \\ \frac{1}{2} + \frac{t^2}{2} & \frac{1}{2} < t \le 1 \\ 1 & t > 1 \end{cases}$$