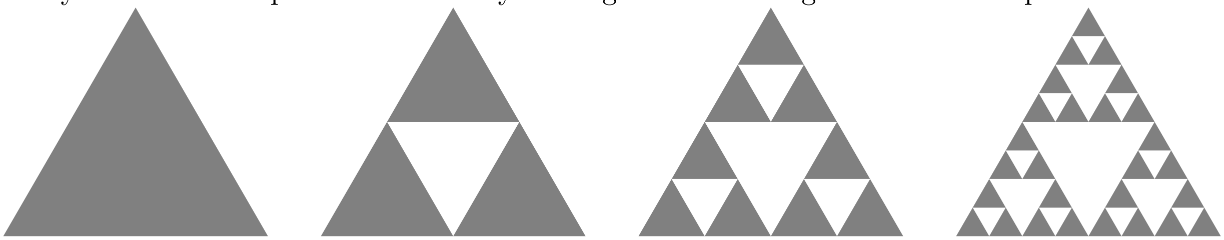
10. [9 points] Molly constructs a pattern using paper in the shape of an equilateral triangle of side length 1.

- **Step 1:** Molly divides the triangle into 4 congruent triangles, and removes the triangle in the middle.
- **Step 2:** For each of the remaining triangles, Molly repeats the previous step.

Molly continues this process indefinitely. A diagram illustrating the first few steps is shown below.



$n = 0$ $n = 1$ $n = 2$ $n = 3$

The area of an equilateral triangle with side length a is given by $\frac{\sqrt{3}}{4}a^2$.

a. [4 points] For $n \geq 1$, let A_n be the **total** area of all the triangles Molly has **removed** after n steps. For example, $A_1 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2$. Find A_2 and A_3. You do not need to simplify your expressions.

Solution: We see that

$$A_2 = A_1 + 3 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 + \frac{3}{4}\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right) = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4}\right)$$

and

$$A_3 = A_2 + 3^2 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{2^3}\right)^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 + \frac{3}{4}\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right) + \left(\frac{3}{4}\right)^2\left(\frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2\right)$$
$$= \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right)$$

Answer: $A_2 = \underline{\quad \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4}\right) \quad}$

Answer: $A_3 = \underline{\quad \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2\right) \quad}$

b. [4 points] Find a **closed-form** expression for A_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive.

Solution: Following the pattern in part **a.**, we have

$$A_n = \frac{\sqrt{3}}{16}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^n\right) = \frac{\sqrt{3}}{16}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k$$

Using the formula for the sum of a finite geometric series:

$$A_n = \frac{\sqrt{3}}{16}\sum_{k=0}^{n-1}\left(\frac{3}{4}\right)^k = \frac{\sqrt{3}}{16}\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

Answer: $A_n = \dfrac{\sqrt{3}}{16}\left(\dfrac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}\right)$

c. [1 point] Find the limit $\lim\limits_{n\to\infty} A_n$.

Answer: $\lim\limits_{n\to\infty} A_n = \dfrac{\sqrt{3}}{4}$