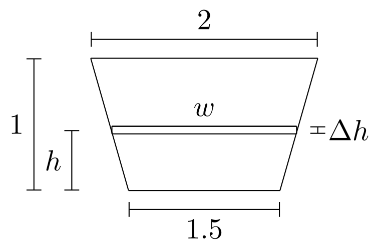
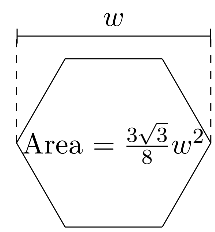
8. [11 points] Rachel is relaxing in her custom hexagonal hot tub when the gemstone from her engagement ring, a cherished gift from William, slips off and sinks to the bottom. Desperate to retrieve it, she decides to pump all the foaming water out over the top edge of the tub.

As viewed from the side, the hot tub is in the shape of a trapezoid, as depicted on the left below, and its horizontal cross-sections are regular hexagons. The diameter of the hot tub is 2 meters at the top rim and 1.5 meters at the bottom, and the tub has a height of 1 meter.

In this problem, you may use the fact that the area of a regular hexagon with diameter w is given by $A = \frac{3\sqrt{3}}{8}w^2$, and that the acceleration due to gravity is $g = 9.8$ m/s^2.



Side view of hot tub



Hexagonal cross-section with diameter w

a. [2 points] Let w be the diameter of a thin horizontal slice of water at a height h meters above the bottom of the tub. Find a formula for w in terms of h.

Solution: Observe that w is a linear function of h with slope 0.5 and $w(0) = 1.5$:

$$w = 1.5 + 0.5h$$

Answer: $w =$ $1.5 + 0.5h$

b. [5 points] Suppose the density of foaming water at h meters above the bottom of the tank, in kg/m^3, is given by the function $\rho(h)$. Write an expression which approximates the **weight** of a horizontal slice of the foaming water which is h meters above the bottom of the tank, and which has small thickness Δh. Your expression may include the letters h, w, and/or ρ, but should not contain any integrals. Include units.

Solution: The volume of a horizontal slice of the foaming water at height h is given by

$$\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h.$$

The approximate weight of the slice is obtained by multiplying by the density, and multiplying the resultant mass by the acceleration due to gravity, giving

$$9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h.$$

Answer: $9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5 + 0.5h)^2\Delta h$ **Units:** N

c. [4 points] The tub is initially filled with water to a height of 0.8m above the **bottom** of the tub. Write an expression involving an integral for the total work needed to pump all of the water over the top edge of the hot tub. Your expression may include the letters h, w, and/or ρ. Include units.

Solution: The approximate work required to lift a slice of water at height h over the top of the hot tub is given by multiplying its weight by the distance traveled:

$$9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\Delta h$$

Integrate from $h=0$ to $h=0.8$ to get the total work done:

$$\text{Total Work} = \int_0^{0.8} 9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\;dh$$

Answer: $\int_0^{0.8} 9.8\rho(h)\frac{3\sqrt{3}}{8}(1.5+0.5h)^2(1-h)\;dh$ **Units:** J