4. (8 points) (a) Give the Taylor series about the point $t = 0$ of the function

$$f(t) = \frac{\sin t}{t}.$$  

(You are allowed to use the standard Taylor series expansions without deriving them).

We know that

$$\sin(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}.$$  

so dividing by $t$ gives

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{6} + \frac{t^4}{120} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}.$$

(b) Is the following statement “True” or “False”? Explain why, if true, or why not, if false.

The Taylor series about the point $x = 0$ of the sine integral function, defined by

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt,$$

is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots.$$

True. False.

The series in part (a) can be integrated term by term to get the Taylor series of the Sine Integral function. That is

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} \, dt = \int_0^x 1 \, dt - \int_0^x \frac{t^2}{6} \, dt + \int_0^x \frac{t^4}{120} \, dt - \cdots = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots$$  

$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!} \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}.$$