4. (8 points) **(a)** Give the Taylor series about the point $t = 0$ of the function

$$f(t) = \frac{\sin t}{t}.$$

(You are allowed to use the standard Taylor series expansions without deriving them).

We know that

$$\sin(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}$$

so dividing by t gives

$$\frac{\sin(t)}{t} = 1 - \frac{t^2}{6} + \frac{t^4}{120} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n+1)!}.$$

(b) Is the following statement "True" or "False"? Explain why, if true, or why not, if false.

The Taylor series about the point $x = 0$ of the sine integral *function, defined by*

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt,$$

is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} = x - \frac{x^3}{18} + \frac{x^5}{600} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots.$$

$\boxed{\text{True.}}$ False.

The series in part (a) can be integrated term by term to get the Taylor series of the Sine Integral function. That is

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt = \int_0^x 1\, dt - \int_0^x \frac{t^2}{6}\, dt + \int_0^x \frac{t^4}{120}\, dt - \cdots = x - \frac{x^3}{18} + \frac{x^5}{600} - \ldots$$
$$= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!}\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}.$$