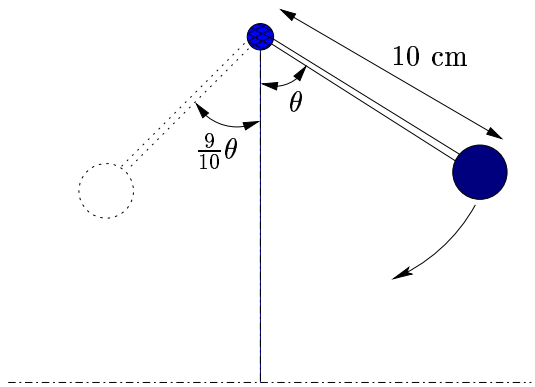
8. (12 points) You begin a pendulum swinging in the position shown in the figure below with $\theta = \pi/4$. Assume the pendulum travels in a circular arc, swinging to the left past the center line shown in the figure and then returning to the right. Notice that the pendulum must briefly stop its motion before it can change direction. We define one "swing" of the pendulum to be the motion between the times when the pendulum stops its motion to change direction. For example, the first "swing" is the motion from the time you release the pendulum until it swings all the way to the left. The second "swing" is the motion coming back from the left to the right, and so on.



(a) Assume that at the end of each swing the pendulum makes an angle of $\frac{9}{10}$ the angle it made when it began the swing. What angle does the pendulum make after its second swing? After its third swing? After its nth swing?

If $\theta_j > 0$ is the angle made by the pendulum with the vertical by the pendulum after the j-th swing, then $\theta_0 = \pi/4$ (given), $\theta_1 = .9\pi/4$, $\theta_2 = .9\theta_1 = (.9)^2\pi/4$, and in general $\theta_{j+1} = .9\theta_j$. Therefore,

$$\theta_2 = (.9)^2\frac{\pi}{4}, \qquad \theta_3 = (.9)^3\frac{\pi}{4}, \qquad \theta_n = (.9)^n\frac{\pi}{4}.$$

(b) Recall that the arc length of a circle is given by the formula $s = r\alpha$ where s is arc length, r is the radius of the circle, and α is the angle measuring the arc length. How far does the weight travel on its first swing? On its second swing? On its nth swing?

If S_j = distance travelled on the j-th swing, then

$$\begin{aligned}
S_1 &= \textit{radius} \times \textit{angle} = 10(\theta_0 + \theta_1) = 10\left(\frac{\pi}{4} + (.9)\frac{\pi}{4}\right) = 19\frac{\pi}{4}\ cm \\
S_2 &= 10(\theta_1 + \theta_2) = 10\left((.9)\frac{\pi}{4} + (.9)^2\frac{\pi}{4}\right) = (.9)S_1\ cm \\
S_n &= 10(\theta_{n-1} + \theta_n) = 10\left((.9)^{n-1}\frac{\pi}{4} + (.9)^n\frac{\pi}{4}\right) = (.9)\left((.9)^{n-2}\frac{\pi}{4} + (.9)^{n-1}\frac{\pi}{4}\right) = (.9)S_{n-1} \\
&= (.9)^2 S_{n-2} = (.9)^3 S_{n-3} = \ldots (.9)^{n-1}S_1 = (.9)^{n-1}(19)\frac{\pi}{4}\ cm
\end{aligned}$$

Problem continued on next page.

(c) What is the total distance the weight has travelled after 30 swings?

Using part (b), this number is

$$S_1 + S_2 + \ldots S_{30} = S_1 + (.9)S_1 + (.9)^2 S_1 + \cdots + (.9)^{29} S_1 = S_1 \left(1 + (.9) + (.9)^2 + (.9)^{29}\right)$$
$$= S_1 \frac{1 - (.9)^{30}}{1 - .9} = 10 S_1 (1 - (.9)^{30})$$

where the next to last equality is obtained using the formula for the sum of a finite geometric series, $1 + x + x^2 + \cdots + x^{n-1} = (1 - x^n)/(1 - x)$. *Using a calculator to compute* $1 - (.9)^{30})$ *and* $S_1 = 19\frac{\pi}{4}$, *we find that the distance travelled in 30 swings is, correct to the digits shown,* $\left(190\frac{\pi}{4}\right) .9576 = 142.8998$ *cm.*

(d) If the pendulum were allowed to swing forever how far would it travel?

This is similar to part (c) except that we must compute the sum of the infinite geometric series, for which we have a known formula, $1 + x + x^2 + \cdots = 1/(1 - x)$ *whenever* $|x| < 1$.

$$S_1 + S_2 + S_3 + \cdots + S_n + \ldots = S_1 \left(1 + (.9) + (.9)^2 + (.9)^3 + \cdots + (.9)^n + \ldots\right)$$
$$= S_1 \frac{1}{1 - .9} = 10 S_1 = 190 \frac{\pi}{4} \approx 149.2257\, cm.$$

More than 95% of the total distance travelled would be taken up in the first 30 swings.