9. (15 points) Frodo Baggins of the Shire is given the task of taking the ring of power from the elven kingdom of Rivendell to Mount Doom, 100 km away, to destroy it. The ring's weight w (in kg) grows at a rate of one-one-hundredth (1/100) of Frodo's distance x (in km) from Rivendell as Frodo proceeds on his journey. Frodo can travel toward Mount Doom at the rate of 2.5 km per hour except that the weight of the ring of power slows his rate of travel (in km/hr) to Mount Doom by one-twentieth (1/20) of the weight of the ring (in kg). Suppose that the ring weighs .001 kg (1 gram) when he begins his journey from Rivendell.

(a) Write a pair of differential equations that model this situation. What are the initial conditions at time $t = 0$?

With $w(t)$, the weight of the ring t hours after leaving Rivendell and $x(t)$, the distance from Rivendell on the road to Mound Doom,

$$\begin{array}{ll} \dfrac{dw}{dt} = \dfrac{x}{100}, & \dfrac{dx}{dt} = 2.5 - \dfrac{w}{20} \\ w(0) = .001 & x(0) = 0 \end{array}$$

(b) What differential equation models the relationship between the weight of the ring and the distance from Rivendell?

$$\frac{dx}{dw} = \frac{\frac{dx}{dt}}{\frac{dw}{dt}} = \frac{2.5 - \frac{w}{20}}{\frac{x}{100}} = \frac{250 - 5w}{x}.$$

(c) Find the trajectory that Frodo follows in the x-w phase plane by solving the differential equation you found in part (b). (Show your work. Don't forget the initial conditions from part (a)).

We solve the differential equation in part (b) for the trajectories of the solutions by separation of variables. One has $x\,dx = (250 - 5w)dw$. Integrate to get $x^2/2 = (250 - 5w)^2/(-10) + C$ where C is a constant of integration. Simplifying the equation then gives

$$x^2 + 5(50 - w)^2 = A.$$

where $A = 2C$. Since $x = 0, w = .001$ is a point on the trajectory, we can find the constant A by substituting these values into the equation. We find $A = 5(50 - .001)^2 \simeq 5(50)^2$. so the trajectory is

$$x^2 + 5(50 - w)^2 = 5(49.999)^2.$$