11. (13 points) In normal conditions, the thyroid hormone (Hormone T), produced in the thyroid gland, and the thyroid successor hormone (Hormone S), produced in the pituitary gland, form a so-called "auto-regulated feedback process". The amount of one in the bloodstream influences the production of the other, and vice-versa. The simple system given below models this process, where x is the amount of Hormone T (in standard units), and y is the amount of Hormone S (in standard units), present in the bloodstream at time t hours.

$$\frac{dx}{dt} = 3 - y\,, \qquad \frac{dy}{dt} = x - 2\,.$$

(a) Find all equilibrium solutions (if any) of the system.

To find the equilibrium solutions of a system of differential equations, we must solve the equations $dx/dt = 0$ and $dy/dt = 0$ simultaneously for x and y. For our system, doing so immediately shows that there is only one equilibrium solution, namely $\mathbf{x(t) = 2}$ and $\mathbf{y(t) = 3}$.

(b) Suppose that at $t = 0$, the amount of Hormone T in the blood was 1.0 and the amount of Hormone S was 3.5, both in standard units. Find the equation of the trajectory of the corresponding solution curve in the phase plane. *Show your work.*

We first convert the system of two equations into a single differential equation linking y and x together without the variable t. That equation is:

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{x-2}{3-y}\,.$$

The given initial conditions becomes the single condition $y(1.0) = 3.5$.

Separating the variables produces

$$(3-y)\,dy = (x-2)\,dx\,.$$

We integrate both sides of this last equation to discover the relationship

$$-\frac{1}{2}(3-y)^2 + C = \frac{1}{2}(x-2)^2\,,$$

where C is an arbitrary integration constant whose value may be determined precisely by using the condition $y(1.0) = 3.5$. Doing so yields

$$-\frac{0.5^2}{2} + C = \frac{(-1)^2}{2} \quad\Longrightarrow\quad C = \frac{5}{8}\,.$$

Therefore, the equation of the trajectory with the given initial conditions is

$$\mathbf{(y-3)^2 + (x-2)^2 = \frac{5}{4}}\,.$$

You may recognize this equation as that of the circle centered at the point $(2,3)$ of radius $\sqrt{5}/2$. One should not be a surprised to find the trajectory to be a circle. Indeed, we are told that in normal conditions, the hormonal pair is auto-regulated. This means that the amount of each hormone in the bloodstream is periodic in time. We will see in the second part of this problem that when the system is deregulated, the solution curve is no longer a circle.

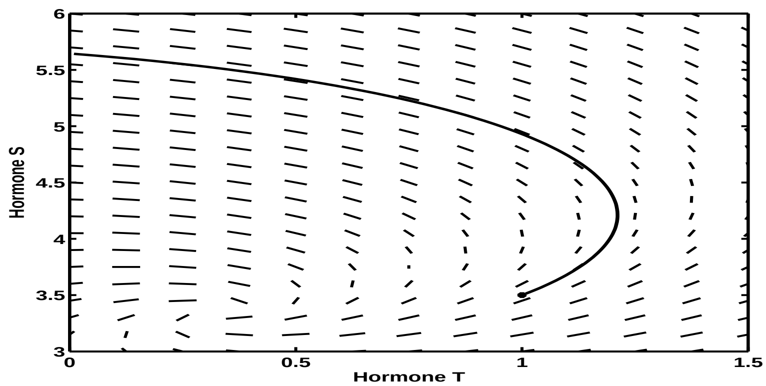
Solution continued on next page.

Solution continued from previous page.

If the patient's diet lacks iodine (e.g. from salt), the chemical agent responsible for detecting the presence of Hormone S in the blood is no longer active. The above model must be replaced by the new system:

$$\frac{dx}{dt} = 3 - y + x\,, \qquad \frac{dy}{dt} = x - 2 + \frac{y}{2}\,.$$

The slope field for the differential equation that describes the trajectories of this system is shown on the figure below.



(c) Sketch on the figure the trajectory corresponding to the initial values in part **(b)**; that is, $x(0) = 1.0$ and $y(0) = 3.5$. *You need not solve any differential equation.*

To sketch the trajectory, we first locate the initial point which we are told has coordinates $(1.0, 3.5)$. *Then we have to decide which direction the curve follows. Substituting* $x = 1.0$ *and* $y = 3.5$ *in the new system, we find* $dx/dt = 1.5$ *and* $dy/dt = 0.75$*, which are both positive. Accordingly, the trajectory moves in the positive* x *and positive* y *directions. Following the slope field in that direction from the initial point, we arrive at the curve shown above.*

(d) In the context of this problem, briefly describe how the amounts of the hormones change from their initial values as time increases.

For a short while, roughly up to the point $(1.2, 4.2)$*, both hormones increase in amount. Probably, the hormonal system is not totally deregulated yet. Things turn bad past the point* $(1.2, 4.2)$. *The amount of Hormone T continually decreases, while the amount of Hormone S continually increases. This pattern continues until the trajectory hits the y-axis, roughly at the point* $(0.0, 5.6)$.

The hormonal system is totally deregulated. The presence of Hormone S is no longer detected, and the body fills this apparent lack by producing more and more of it. In turn, Hormone S inhibits the production of Hormone T whose amount decreases until none is left in the blood.

The model given here, although greatly simplified, depicts an actual medical condition known as "goiter": the thyroid gland inflates due the large quantity of Hormone S stagnating in it.