9. (10 points) At age 65, Mrs. Smith retires with \$1,000,000 in her retirement account. Assume that after retirement:

(i) She receives interest of 5% per year (compounded continuously) on the balance in the account, and this money is reinvested in the account ;

(ii) She withdraws money (for living expenses) from the account at a continuous rate of \$60,000 per year.

(a) Write the initial value problem for the balance $B(t)$ of dollars in the account t years after Mrs. Smith retires.

From the information given in the problem, we deduce that the balance in dollars, $B(t)$, in Mrs. Smith's account satisfies the differential equation

$$B'(t) \;=\; 0.05\,B(t) \;-\; 60,000$$

and has the initial value $B(0) = 1,000,000$ dollars at time 0.

(b) Will Mrs. Smith ever exhaust the retirement account, i.e. reduce the balance in the account to zero? *Explain.*

The amount of money B in the account is being reduced at all times. Indeed, originally $dB/dt = 0.05B - 60,000 < 50,000 - 60,000 < 0$. As B decreases, so does $0.05B$. Hence this also happens at all future times. In particular, the account decreases by at least \$10,000 each year, and it will thus eventually be depleted.

Although the problem does not require it, one can find out when the account will be depleted by solving the equation $B(T) = 0$, for T. The solution to the differential equation in part **(a)** *is $B(t) = 1,200,000 + A\,e^{0.05t}$, where A is a constant. Since $B(0) = 1,000,000$, we get $A = -200,000$ and thus $B(t) = 1,200,000 - 200,000\,e^{0.05t}$. The solution of the equation $B(T) = 0$ is found to be $T = (\ln 6)/0.05 \approx 36$ so it will take about 36 years to deplete the account.*

(c) Are there any equilibrium solutions to the differential equation of part **(a)**? If so, explain their meaning in terms of Mrs. Smith's money.

The only equilibrium solution of the differential equation from part **(a)** *is where $dB/dt = 0$ for all times t or where $B = 60,000/0.05 = 1,200,000$.*

This is the amount of money necessary so that the interest generated is exactly equal to the amount that Mrs. Smith is withdrawing, so that the balance in the account remains constant.