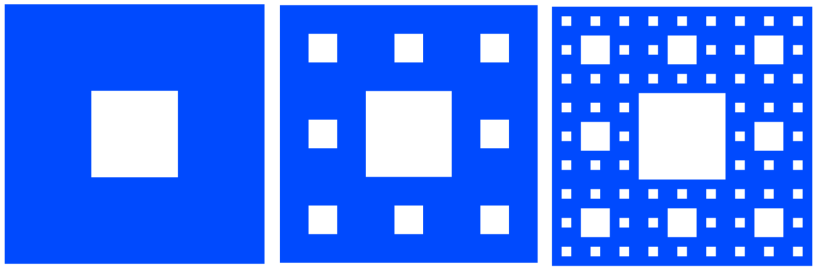
4. (8 points) The Sierpinski Carpet is an example of a mathematical object called a fractal. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure below shows the first three steps of the construction.)



At the n-th step of the process, 8^{n-1} squares are removed, each with area $\left(\frac{1}{9}\right)^n$. Thus, the area removed at the n-th step is $A_n = \left(\frac{8^{n-1}}{9^n}\right)$. There are infinitely many steps in the process.

(a) (2 pts.) Find the limit of the sequence $A_1, A_2, A_3, \ldots$.

$$\lim_{n\to\infty}\left(\frac{8^{n-1}}{9^n}\right) = \frac{1}{9}\lim_{n\to\infty}\left(\frac{8}{9}\right)^n = 0, \qquad \text{as } 8/9 < 1.$$

(b) (2 pts.) Write a mathematical expression that represents A, the *total* sum of the areas of the removed squares after infinitely many steps of the process.

$$A = A_1 + A_2 + A_3 + \cdots = \sum_{n=1}^{\infty}\left(\frac{8^{n-1}}{9^n}\right) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n.$$

(c) (4 pts.) Exactly how much area is removed in all? Show your work.

Since the infinite sum

$$A = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n$$

is a geometric series with ratio $8/9 < 1$, we have:

$$A = (1/9)\,\frac{1}{1-8/9} = (1/9)\;9 = 1.$$

So, the total area removed from the carpet is equal to 1.