8. (9 points) Recall that the Taylor series for $\cos x$ about $x = 0$ is given by $1 - x^2/2! + x^4/4! - x^6/6! + \cdots$.

(a) (5 pts.) The Taylor series for $\cos x$ equals $\cos x$ wherever it converges. For which x-values does the Taylor series for $\cos x$ equal the function $\cos x$? Give a precise step-by-step argument that justifies your answer. *No graphs are allowed as justification.*

For the given series we have:

$$a_n = (-1)^n \frac{x^{2n-2}}{(2n-2)!}, \qquad a_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)-2}}{(2(n+1)-2)!} = (-1)^{n+1} \frac{x^{2n}}{(2n)!},$$

so then we get that

- $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{2n}}{(2n)!} \frac{(2n-2)!}{x^{2n-2}} = \frac{x^2}{(2n)(2n-1)}, \qquad \text{and}$$
- $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = x^2 \lim_{n\to\infty} \frac{1}{(2n)(2n-1)} = 0.$$

So the radius of convergence is infinite, or convergence holds for *all* x-values (that is, all real numbers x.)

(b) (4 pts.) Find all the solutions to the equation

$$1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots = 0.$$

You must show your work clearly and give exact answers. Calculator approximations or methods will receive no credit.

We have,

- $$1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots = \cos(3x) \qquad \text{and}$$
- $\cos(3x) = 0$, which means: $3x = \pi/2 + k\pi$,

or

$$x = \frac{\pi}{6} + k\frac{\pi}{3}, \quad \text{where } k \text{ is any integer.}$$