3. [7 points] The function erf(x) is defined to be $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$. Find the Taylor series for erf(x) around $x = 0$.

Solution:
We know that the Taylor series for e^t at $t = 0$ is $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$. Thus the series for $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n}}{n!}$. We can integrate this to find the Taylor series for erf(x):

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n}}{n!}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n\, t^{2n}}{n!}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)\,n!}.$$

Note that if we try and derive this from first principles it is more difficult to get the general term: $\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$, so $\mathrm{erf}''(x) = \frac{2}{\sqrt{\pi}}(-2xe^{-x^2})$, and so on: $\mathrm{erf}'''(x) = \frac{2}{\sqrt{\pi}}(-2e^{-x^2} + 4x^2e^{-x^2})$, $\mathrm{erf}^{4}(x) = \frac{2}{\sqrt{\pi}}(12xe^{-x^2} - 8x^3e^{-x^2})$, etc. So $\mathrm{erf}(0) = 0$, $\mathrm{erf}'(0) = \frac{2}{\sqrt{\pi}}$, $\mathrm{erf}''(0) = 0$, $\mathrm{erf}'(0) = \frac{2}{\sqrt{\pi}} \cdot (-2)$, $\mathrm{erf}^{4}(0) = 0$, etc. This gives us the first two non-zero terms of the series, but doesn't shed much insight on the general progression.