5. [10 points] A mathematician proposed* that the velocity, $v(t)$, of a sprinter running less than 300 meters might satisfy the differential equation $\frac{dv}{dt} = k(v - R)$, for some constants k and R. For a sprint, it makes sense that $v(0) = 0$.

(a) [4 points of 10] Find the general solution to this differential equation.

Solution:
We can separate variables to get $\frac{dv}{v-R} = kdt$. Integrating both sides, we have $\ln|v - R| = kt + C$, so that $v = R + Ae^{kt}$, for some arbitrary constant $A = \pm e^C$.

(b) [2 points of 10] Find the particular solution to the initial value problem. (Your answer may involve the constants k and R.)

Solution:
We know that $v(0) = 0$. This gives $0 = R + A$, so $A = -R$, and the particular solution is $v = R(1 - e^{kt})$.

(c) [4 points of 10] Linford Christie won the men's 100 meter race in the 1993 World Track Championships in a time of 9.87 sec. If one second into the race he had reached 51% of his maximum possible speed, find values for the parameters k and R in the problem.

Solution:
Note that $v = R$ is an equilibrium solution to the equation, and that as $t \to \infty$ we must have $v \to R$ (k is negative, so that $\frac{dv}{dt} > 0$ when $v = 0$). Thus Christie's top speed is R, and if Christie reaches 51% of his maximum speed after one second, we know $v(1) = 0.51R$. Thus $R(1 - e^k) = 0.51R$, so that $e^k = 0.49$, and $k = \ln(0.49) \approx -0.713$. As we expected, this is less than zero. Then we know that in 9.87 seconds Christie covers 100 m. This is the integral of the velocity, so that

$$\begin{aligned} 100 = \int_0^{9.87} R(1 - e^{-0.713t})\,dt &= R\left(t + \frac{1}{0.713}\,e^{-0.713t}\right)\Bigg|_0^{9.87} \\ &= R\left(9.87 + 1.403\left(e^{-7.037} - 1\right)\right) \\ &= 8.468R. \end{aligned}$$

Dividing by 8.468, we find $R = \frac{100}{8.468} = 11.81$.

* *J.B. Keller, in Physics Today, Sept. 1973*