9. [12 points] It turns out that students at Alex and Chris' university have a strong tradition of taking university math classes. In fact, Chris determines that for the function $p(t) = \frac{1}{5(\frac{1}{5}+t)^2}$, the fraction of students having completed between t and $t + \Delta t$ years of collegiate mathematics is given approximately by $p(t)\,\Delta t$.

(a) [4 points of 12] Carefully find the fraction of students who have completed at least two years of university mathematics.

Solution:
Given the property that $p(t)\,\Delta t$ gives the fraction of students having completed between t and $t+\Delta t$ years of collegiate mathematics, we can find the fraction having completed at least two years of mathematics by integrating. This is $\int_2^\infty \frac{1}{5(\frac{1}{5}+t)^2}\,dt$. This is clearly an improper integral, so we evaluate it with some care and a limit. $\int_2^\infty \frac{1}{5(\frac{1}{5}+t)^2}\,dt = \lim_{b\to\infty} \int_2^b \frac{1}{5(\frac{1}{5}+t)^2}\,dt = \lim_{b\to\infty} (-\frac{1}{5(\frac{1}{5}+b)} + \frac{1}{5(\frac{1}{5}+2)}) = \frac{1}{11}$. Or, about 9%.

(b) [4 points of 12] Let $q(x)$ be the fraction of students that complete no more than x years of university mathematics. Write an integral that gives $q(x)$. Then evaluate your integral to find a formula for $q(x)$.

Solution:
We note that $q(x) = \int_0^x p(t)\,dt$, an antiderivative of $p(t)$. Evaluating, we get $q(x) = 1 - \frac{1}{5(\frac{1}{5}+x)} = 1 - \frac{1}{1+5x} = \frac{5x}{1+5x}$.

(c) [4 points of 12] We might think that the integral $\int_0^\infty t\,p(t)\,dt$ would give the average number of years of university mathematics that the students take. Explain why this does not make sense in this context. (Hint: how large is this value?)

Solution:
Note that for $t \geq 1$, $\frac{t}{5(\frac{1}{5}+t)^2} > \frac{t}{5(t+t)^2} = \frac{1}{20t}$, and $\int_1^\infty \frac{1}{20t}\,dt$ diverges. Thus $\int_1^\infty \frac{t}{5(\frac{1}{5}+t)^2}\,dt$ diverges, which means that $\int_0^\infty \frac{t}{5(\frac{1}{5}+t)^2}\,dt$ must also. This suggests that the mean number of years of university mathematics that the students study is infinite, which seems unlikely.