9. [12 points] It turns out that students at Alex and Chris’ university have a strong tradition of taking university math classes. In fact, Chris determines that for the function \( p(t) = \frac{1}{5(t + t)}^2 \), the fraction of students having completed between \( t \) and \( t + \Delta t \) years of collegiate mathematics is given approximately by \( p(t) \Delta t \).

(a) [4 points of 12] Carefully find the fraction of students who have completed at least two years of university mathematics.

Solution:
Given the property that \( p(t) \Delta t \) gives the fraction of students having completed between \( t \) and \( t + \Delta t \) years of collegiate mathematics, we can find the fraction having completed at least two years of mathematics by integrating. This is
\[
\int_2^\infty \frac{1}{5(t + t)}^2 \, dt
\]
This is clearly an improper integral, so we evaluate it with some care and a limit.
\[
\int_2^\infty \frac{1}{5(t + t)}^2 \, dt = \lim_{b \to \infty} \int_2^b \frac{1}{5(t + t)}^2 \, dt = \lim_{b \to \infty} \left( \frac{1}{5(t + t)} + \frac{1}{5(t + t)} \right) = \frac{1}{11}. \text{ Or, about 9%}. 
\]

(b) [4 points of 12] Let \( q(x) \) be the fraction of students that complete no more than \( x \) years of university mathematics. Write an integral that gives \( q(x) \). Then evaluate your integral to find a formula for \( q(x) \).

Solution:
We note that \( q(x) = \int_0^x p(t) \, dt \), an antiderivative of \( p(t) \). Evaluating, we get \( q(x) = 1 - \frac{1}{5(x + x)} = 1 - \frac{5x}{1 + 5x} \).

(c) [4 points of 12] We might think that the integral \( \int_0^\infty t p(t) \, dt \) would give the average number of years of university mathematics that the students take. Explain why this does not make sense in this context. (Hint: how large is this value?)

Solution:
Note that for \( t \geq 1 \), \( \frac{t}{5(t + t)}^2 > \frac{t}{5(t + t)}^2 = \frac{1}{5t} \), and \( \int_1^\infty \frac{1}{5t} \, dt \) diverges. Thus \( \int_1^\infty \frac{t}{5(t + t)}^2 \, dt \) diverges, which means that \( \int_0^\infty \frac{t}{5(t + t)}^2 \, dt \) must also. This suggests that the mean number of years of university mathematics that the students study is infinite, which seems unlikely.