5. [12 points] Solve the following:

(a) [4 points of 12] Explain how, by starting with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \qquad |x| < 1,$$

you can derive the Taylor series for $\ln(1+x)$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$. (Your explanation need not be a step-by-step derivation, but should clearly indicate what steps are necessary to complete it.)

(b) [4 points of 12] If money is invested at an interest rate r, compounded monthly, it will double in n years, where n is given by

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}.$$

(This can be deduced from the formula $(1+\frac{r}{12})^{12n} = 2$, but we do not need this derivation for this problem.) Use the Taylor polynomial of degree 1 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time n is approximately proportional to $\frac{1}{r}$, and find the constant of proportionality, k.

(c) [4 points of 12] Use the Taylor polynomial of degree 2 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time $n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}$ may be approximated by an expression of the form $\frac{k}{r-ar^2}$. Find k and a.