5. [12 points] Solve the following:

(a) [4 points of 12] Explain how, by starting with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \qquad |x| < 1,$$

you can derive the Taylor series for $\ln(1+x)$, $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$. (Your explanation need not be a step-by-step derivation, but should clearly indicate what steps are necessary to complete it.)

Solution:
Substituting $-x$ for x in the geometric series, we obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

so that, integrating term-by-term, we have

$$\ln(1+x) = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

We know that $\ln(1+0) = \ln(0) = 0$, so $C = 0$.

(b) [4 points of 12] If money is invested at an interest rate r, compounded monthly, it will double in n years, where n is given by

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}.$$

(This can be deduced from the formula $(1+\frac{r}{12})^{12n} = 2$, but we do not need this derivation for this problem.) Use the Taylor polynomial of degree 1 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time n is approximately proportional to $\frac{1}{r}$, and find the constant of proportionality, k.

Solution:
The Taylor polynomial of degree one, $P_1(x)$, is just the Taylor series truncated at the linear term: $P_1(x) = x$. Thus $\ln(1+\frac{r}{12}) \approx \frac{r}{12}$ for small r, and

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})} \approx \frac{\ln(2)}{12}\frac{1}{\frac{r}{12}} = \frac{\ln(2)}{r}.$$

We see that n is proportional to $\frac{1}{r}$, and the constant of proportionality is $k = \ln(2)$.

(c) [4 points of 12] Use the Taylor polynomial of degree 2 for $\ln(1+x)$ near $x = 0$ to show that for small r the doubling time $n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})}$ may be approximated by an expression of the form $\frac{k}{r-ar^2}$. Find k and a.

Solution:
The Taylor polynomial of degree 2, $P_2(x)$, is just the Taylor series truncated at the quadratic term: $P_2(x) = x - \frac{1}{2}x^2$. Thus $\ln(1+\frac{r}{12}) \approx \frac{r}{12} - \frac{1}{2}(\frac{r}{12})^2$, so

$$n = \frac{\ln(2)}{12}\frac{1}{\ln(1+\frac{r}{12})} \approx \frac{\ln(2)}{12}\frac{1}{\frac{r}{12} - \frac{1}{2}(\frac{r}{12})^2} = \frac{\ln(2)}{r - \frac{1}{24}r^2}.$$

We therefore have $k = \ln(2)$, as we found before, and $a = \frac{1}{24}$.